

Deformed phase space for 3d loop gravity and hyperbolic discrete geometries

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We revisit the loop gravity space phase for 3D Riemannian gravity by algebraically constructing the phase space $T^*\text{SU}(2) \sim \text{ISO}(3)$ as the Heisenberg double of the Lie group $\text{SO}(3)$ provided with the trivial cocycle. Tackling the issue of accounting for a non-vanishing cosmological constraint $\Lambda \neq 0$ in the canonical framework of 3D loop quantum gravity, $\text{SL}(2, \mathbb{C})$ viewed as the Heisenberg double of $\text{SU}(2)$ provided with a non-trivial cocycle is introduced as a phase space. It is a deformation of the flat phase space $\text{ISO}(3)$ and reproduces the latter in a suitable limit. The $\text{SL}(2, \mathbb{C})$ phase space is then used to build a new, deformed LQG phase space associated to graphs. It can be equipped with a set of Gauss constraints and flatness constraints, which form a first class system and Poisson-generate local 3D rotations and deformed translations. We provide a geometrical interpretation for this lattice phase space with constraints in terms of consistently glued hyperbolic triangles, i.e. hyperbolic discrete geometries, thus validating our construction as accounting for a constant curvature $\Lambda < 0$. Finally, using ribbon diagrams, we show that our new model is topological.

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Introduction

3D gravity is an interesting laboratory to explore the quantization of gravity. It retains all the main features of 4d gravity but is much simpler since it is a topological field theory and admits a finite number of (topological) degrees of freedom [1], which can be solved and quantized exactly. In the main different quantization frameworks, Chern-Simons, spinfoam and loop quantum gravity (LQG), the cosmological constant Λ is taken as a coupling constant on the same grounds as Newton's constant for gravity G_N .

When $\Lambda = 0$, all of these frameworks can be used to describe the quantum gravity regime. It is possible to construct bridges between each of them so that they are indeed consistent [2–5]. An interesting feature is that in all three frameworks, a quantum group -the Drinfeld double- appears as the natural symmetry structure [2, 4].

When $\Lambda \neq 0$, things get more intricate. The Chern-Simons and spinfoam approaches show that the use of a quantum group is necessary to encode $\Lambda \neq 0$ [6, 7]. Furthermore both approaches can be related explicitly [8]. Note that in the cases of a Lorentzian signature or a negative cosmological constant $\Lambda < 0$, these two approaches do not yet provide a definite answer since the representations of the associated non-compact quantum groups are not yet entirely understood. The LQG case is rather different and trickier. Indeed Λ appears as a coupling constant in the quantum Hamiltonian constraint, and one does not know how to solve it on the kinematical Hilbert space if $\Lambda \neq 0$. Moreover, these kinematical states of geometry are standard spin network states, which are not defined in terms of a quantum group. This discrepancy is surprising since one would expect the dynamics of 3D loop quantum gravity for $\Lambda \neq 0$ to be given by the Turaev-Viro spinfoam model (similarly to 3D loop quantum gravity for $\Lambda = 0$ defined by the Ponzano-Regge spinfoam model) and thus described in terms of quantum group spin networks. Nevertheless, observables for 3D LQG were recently defined in terms of a quantum gauge group ($\mathcal{U}_q(\mathfrak{su}(2))$ with q real) and studied in details [10, 11]. This analysis showed that the theory for $\Lambda < 0$ can indeed be interpreted as describing quantum hyperbolic geometries, as one could expect.

The key question is then: *how does such quantum group structure appear in the LQG framework?* On the one hand, we have the classical action for gravity with $\Lambda \neq 0$, with its canonical analysis, constraint algebra and geometrical interpretation as constant curvature geometries. On the other hand, we have a candidate quantum theory defined in terms of a quantum gauge group and associated spin network observables and admitting a good geometrical interpretation. The goal is now to connect the two frameworks. See [9] for an early attempt to tackle this issue. In the present article, we do not fully answer this question yet, but we provide an essential step towards the final answer of this question, by introducing a classical phase space for 3D loop quantum gravity with $\Lambda < 0$ which can be interpreted as a classical representation of a quantum gauge group symmetry and whose quantization is expected to lead to the observables of quantum hyperbolic geometries as introduced in [10, 11].

More precisely, we know very well how to discretize the Hamiltonian formulation of the BF action, with $\Lambda = 0$, using standard lattice gauge theory tools [5, 12, 13]. Such discrete theory can then be quantized to give rise to LQG and the Ponzano-Regge spinfoam model. Hence, we would like to construct the analogue of such discrete theory and deal with discrete hyperbolic/spherical geometries, which upon quantization would lead to spin network defined in terms of quantum groups and hopefully the dynamics given by the Turaev-Viro model.

We provide here such a model. The key technical innovation is the use of the Heisenberg double which describes the general framework for phase spaces based on groups [14, 15]. In particular, using this formalism, one can show that $T^*\text{SU}(2)$ can be described by the Euclidian group ISO(3), equipped with a symplectic structure. This allows to deform

$T^*\text{SU}(2)$ to the case where momentum space (i.e. the flux space) is curved. In this case¹, the relevant phase space becomes $\text{SL}(2, \mathbb{C})$. We can then deform the usual lattice gauge construction using the deformed Heisenberg double. We have constructed a set of two first class constraints, the Gauss constraint which encode the $\text{SU}(2)$ invariance at the vertices of the lattice as well as the flatness constraint which encodes the invariance under some deformed translations (since we are dealing with homogeneously curved geometries). In the $\text{SL}(2, \mathbb{C})$ case, the symmetries are non-trivial in the sense that they are equipped with non-trivial Poisson brackets. The symmetry groups are actually Poisson-Lie groups, which are the classical version of the notion of quantum group [17]. We have showed that the model we have constructed is topological and explored the geometrical meaning of the constraints. Just as in the flat case, the Gauss constraint is geometrically equivalent to the (hyperbolic) cosine law and the flatness constraint is equivalent to evaluating the extrinsic curvature in terms of the dihedral angles. By construction, we have the classical Drinfeld double as symmetry structure, both in the flat case ($\text{ISO}(3)$) and the curved case ($\text{SL}(2, \mathbb{C})$).

The discrete lattice gauge theory we have constructed resembles therefore very much to what we would obtain by discretizing a 3D BF theory with a cosmological constant. However the precise link between such a continuum theory and our discrete theory still has to be identified. The quantization of our model will be discussed in the outlook.

The scheme of the article goes as follows. In Section I, we recall the Heisenberg double formalism and in particular how the standard $T^*\text{SU}(2)$ phase space can be described by the group $\text{ISO}(3)$. We recall then how $\text{SL}(2, \mathbb{C})$ can also be seen as a phase space and how $T^*\text{SU}(2)$ is recovered from it in a suitable limit. In Section II, we recall the notion of Poisson symmetries, which are given here by Poisson Lie groups, the classical version of a quantum group. In particular we explain how the classical Drinfeld double appears as the symmetry structure of the Heisenberg double. In the $\text{SL}(2, \mathbb{C})$ case, a deformed notion of translation is obtained. We also provide the infinitesimal realization of the different symmetries (rotations and (deformed) translations), in terms of Poisson brackets. In Section III, we define the lattice gauge theory model by putting on each edge of the lattice the relevant Heisenberg double. A useful graphical representation for this is to use ribbons instead of lines. We provide the definition of a set of first class constraints and show how they relate to the infinitesimal symmetries discussed in Section II. In Section IV, we describe the geometric meaning of the constraints. In Section V, we show that the discrete models we have defined, either in terms of $\text{ISO}(3)$ or $\text{SL}(2, \mathbb{C})$ are topological. We also provide, in appendix, the details of the phase space structure for $\text{SL}(2, \mathbb{C})$ as well as the proofs for the realization of the infinitesimal symmetries in terms of the Poisson brackets.

I. LQG PHASE SPACE AND ITS DEFORMATION

A. Definitions

A Poisson Lie group is a Lie group equipped with an additional structure, a Poisson bracket satisfying a compatibility condition with the group multiplication. Such a Poisson Lie group is associated with a unique Lie bialgebra, the Lie algebra of the Lie group together with an additional structure, a cocycle δ . The cocycle is derived from the Poisson bracket associated to the Poisson Lie group [16, 17].

Consider the Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ with generators e_i , structure constants α_{ij}^k and cocycle $\delta_{\mathfrak{g}}$ which defines a second family of structure constants β_{jk}^{ij} , such that

$$[e_i, e_j] = \alpha_{ij}^k e_k, \quad \delta_{\mathfrak{g}}(e_k) = \beta_{jk}^{ij} e_i \otimes e_k.$$

The classical double $\mathfrak{d}(\mathfrak{g})$ is a Lie bialgebra defined from the following structures.

- The dual Lie bialgebra \mathfrak{g}^* , with generators f^i . The dual map is built from the bilinear map $\langle f^i, e_j \rangle = \delta_j^i$. The Lie bracket on \mathfrak{g}^* is built from the cocycle on \mathfrak{g} .

$$\langle [f^i, f^j], e_k \rangle = \langle f^i \otimes f^j, \delta(e_k) \rangle \rightsquigarrow [f^i, f^j] = \beta_{jk}^{ij} f^k.$$

The cocycle δ_* on \mathfrak{g}^* is built by duality from the Lie bracket on \mathfrak{g} .

$$\langle \delta_*(f^i), e_j \otimes e_k \rangle = \langle f^i, [e_j, e_k] \rangle \rightsquigarrow \delta_*(f^i) = \alpha_{jk}^i f^j \otimes f^k.$$

$(\mathfrak{g}^*, \delta_*)$ is thus the Lie bialgebra with "interchanged" structure constants with respect to $(\mathfrak{g}, \delta_{\mathfrak{g}})$.

¹ We shall focus for simplicity on the Euclidian case, with $\Lambda < 0$.

- As a Lie algebra $\mathfrak{d}(\mathfrak{g})$ with generators e_i, f_j and commutators

$$[e_i, e_j] = \alpha_{ij}^k e_k, \quad [f^i, f^j] = \beta_k^{ij} f^k, \quad [e_i, f^j] = \beta_j^{ik} e_k - \alpha_{jk}^i f^k.$$

Hence, we have that $\mathfrak{d}(\mathfrak{g}) = \mathfrak{g}^* \bowtie \mathfrak{g}$ as a Lie algebra. The bilinear form constructed above, supplemented by $\langle e_i, e_j \rangle = \langle f^i, f^j \rangle = 0$ is then $\mathfrak{d}(\mathfrak{g})$ -invariant.

- The cocycle structure on $\mathfrak{d}(\mathfrak{g})$ is given by

$$\delta_{\mathfrak{d}}(d) = [d \otimes d, r], \quad d \in \mathfrak{d}, \quad r = \sum f^i \otimes e_i,$$

where $r = \sum f^i \otimes e_i$ is the classical r -matrix. When restricted respectively to \mathfrak{g} and \mathfrak{g}^* , we have

$$\delta_{\mathfrak{d}}(e_i) = -\delta(e_i), \quad \delta_{\mathfrak{d}}(f^i) = \delta_*(f^i).$$

- The r -matrix can be split into the antisymmetric and symmetric parts. We note $r^t \equiv e_i \otimes f^i$.

$$\mathbf{a} = \frac{1}{2}(r - r^t) = \frac{1}{2} \sum_i (f^i \otimes e_i - e_i \otimes f^i), \quad \mathbf{s} = \frac{1}{2}(r + r^t).$$

We shall consider the case where \mathbf{s} is non-degenerate and an invariant of \mathfrak{d} . In this case, we say that the classical double \mathfrak{d} is factorizable and quasi-triangular. Note that \mathbf{s} can be interpreted as a (quadratic) Casimir for \mathfrak{d} .

We can integrate \mathfrak{d} to get the Lie group $\mathcal{D} \sim G^* \bowtie G$. It can be equipped with two typical Poisson structures, characterized by the Poisson bivectors π_{\pm} .

$$\pi_{\pm}(D) = -[\mathbf{a}, D \otimes D]_{\pm}, \quad D \in \mathcal{D}, \quad (1)$$

where $[\cdot, \cdot]_-$ denotes the usual commutator and $[\cdot, \cdot]_+$ the anticommutator.

The group \mathcal{D} equipped with π_- is called the *Drinfeld double* of G . It is a Poisson Lie group and as such is not a symplectic space.

The group \mathcal{D} equipped with π_+ is called the *Heisenberg double*. It is a symplectic space. We can determine the Poisson brackets between the matrix elements of $D \in \mathcal{D}$ in terms of the r -matrix, using the expression of π_+ and the expression of \mathbf{a} in terms of r and r^t . We get

$$\{D \otimes D\}_{\pi_+} = -[\mathbf{a}, D \otimes D]_+ = -r(D \otimes D) + (D \otimes D)r^t, \quad (2)$$

where the subscript π_+ refers to the fact that we are using the Poisson bracket structure coming from the bivector π_+ . Since we are going to deal only with this Poisson bracket in this Section, we shall omit it for simplicity. We will come back to the Drinfeld double structure implemented by π_- in Section II when addressing the symmetries. We use the standard notation $D_1 = D \otimes \mathbf{1}$ and $D_2 = \mathbf{1} \otimes D$.

$$\{D \otimes D\} \equiv \{D_1, D_2\} \equiv \begin{pmatrix} \{D_{11}, D_{11}\} & \{D_{11}, D_{12}\} & \dots & \{D_{12}, D_{11}\} & \{D_{12}, D_{12}\} & \dots \\ \{D_{11}, D_{21}\} & \{D_{11}, D_{22}\} & \dots & \{D_{12}, D_{21}\} & \{D_{12}, D_{22}\} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \{D_{21}, D_{11}\} & \{D_{21}, D_{12}\} & \dots & \{D_{22}, D_{11}\} & \{D_{22}, D_{12}\} & \dots \\ \{D_{21}, D_{21}\} & \{D_{21}, D_{22}\} & \dots & \{D_{22}, D_{21}\} & \{D_{22}, D_{22}\} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

We have constructed the classical double of the Lie algebra \mathfrak{g} , with cocycle δ . However, we could also construct the classical double of \mathfrak{g}^* with cocycle δ_* . In this case, we obtain the Lie algebra $\mathfrak{g} \bowtie \mathfrak{g}^*$ with cocycle generated by the r -matrix $\tilde{r} = \sum e_i \otimes f^i = r^t$. The resulting group D will be the same. The difference from the previous construction is how this group is factorized and the symplectic structure on it. We have now $D \sim G \bowtie G^*$. The symplectic structure is given by

$$\{D_1, D_2\} = -\tilde{r}(D \otimes D) + (D \otimes D)\tilde{r}^t = -r^t(D \otimes D) + (D \otimes D)r. \quad (3)$$

Let us illustrate the full construction for $\mathcal{D} = \text{ISO}(3)$ and $\text{SL}(2, \mathbb{C})$. At the infinitesimal level, we start with the Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ (resp. $\mathfrak{su}(2)$) with generators J_i , (resp. σ_i) satisfying the commutation relations $[J_i, J_j] = \epsilon_{ij}^k J_k$, (resp. $[\sigma_i, \sigma_j] = 2i\epsilon_{ij}^k \sigma_k$). But we consider two possible cocycles $\delta_{\mathfrak{g}}$: $\text{ISO}(3)$ comes from a choice of trivial cocycle for $\mathfrak{so}(3)$ whereas $\text{SL}(2, \mathbb{C})$ emerges as the double for $\mathfrak{su}(2)$ equipped with a non-trivial cocycle. Indeed, a given choice of cocycle entirely determines the associated dual Lie bialgebra \mathfrak{g}^* . The details are given in the next two sections.

$$\left(\begin{array}{c} \mathfrak{g} \\ \updownarrow \langle, \rangle \\ \mathfrak{g}^* \end{array} \quad \begin{array}{c} [\cdot, \cdot] \\ \swarrow \delta \\ \searrow \delta_* \end{array} \right) \longrightarrow \left(\begin{array}{l} \mathfrak{d}_r(\mathfrak{g}) \equiv (\mathfrak{d} = \mathfrak{g}^* \bowtie \mathfrak{g}, r) \longrightarrow (\mathcal{D} = G^* \bowtie G, \{, \}_{\pi_{\pm}}) \\ \mathfrak{d}_{\tilde{r}}(\mathfrak{g}^*) \equiv (\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*, \tilde{r}) \longrightarrow (\mathcal{D} = G \bowtie G^*, \{, \}_{\pi_{\pm}}) \end{array} \right)$$

B. ISO(3) as phase space

We first consider $\mathfrak{so}(3)$ equipped with the simplest cocycle, the trivial cocycle $\delta_0 \equiv 0$. The classical double $\mathfrak{d}_0(\mathfrak{so}(3))$ is then given by the following structures.

- We note E^i the dual of J_i . The \mathfrak{d}_0 -invariant bilinear form is $\langle E^i, J_j \rangle = \delta_j^i$, $\langle J_i, J_j \rangle = \langle E^i, E^j \rangle = 0$, $\forall i, j \in \{1, 2, 3\}$.
- The dual Lie algebra $\mathfrak{so}(3)^*$, generated by the E 's is Abelian. Indeed, $[E^i, E^j] = 0$ since the $\mathfrak{so}(3)$ cocycle δ_0 is trivial. Hence $\mathfrak{so}(3)^* \sim \mathbb{R}^3$. Its associated cocycle is given by the constant structure defined by the Lie algebra structure of $\mathfrak{so}(3)$, that is, $\delta_*(E^k) = \epsilon_{ij}^k E^i \otimes E^j$.
- As a Lie algebra $\mathfrak{d}_0(\mathfrak{so}(3)) \sim \mathfrak{iso}(3) = \mathbb{R}^3 \rtimes \mathfrak{so}(3)$.
- The (coboundary) cocycle $\delta_{\mathfrak{d}_0}$ is generated by the r -matrix $r = \sum_i E^i \otimes J_i$.

We construct now the Heisenberg double \mathcal{D}_{π_+} . It is given by the Lie group $\text{ISO}(3)$ while the Poisson structure is specified by the r -matrix, as in (2). A group element $D \in \text{ISO}(3)$ is characterized by a rotation $\mathbf{R} \in \text{SO}(3)$ and a translation $\mathbf{X} \in \mathbb{R}^3$, with product,

$$(\mathbf{R}_1, \mathbf{X}_1)(\mathbf{R}_2, \mathbf{X}_2) = (\mathbf{R}_1 \mathbf{R}_2, \mathbf{X}_1 + \mathbf{R}_1 \triangleright \mathbf{X}_2), \quad (4)$$

where the rotation \mathbf{R}_1 acts in the vector representation on the 3-vector \mathbf{X}_2 . A general element $D = (\mathbf{R}, \mathbf{X})$, can be factorized into a pure translation and a pure rotation.

$$D = (\mathbf{R}, \mathbf{X}) = \ell u, \quad \ell = (\mathbf{1}, \mathbf{X}), \quad u = (\mathbf{R}, \mathbf{0}). \quad (5)$$

The Poisson bracket structure for D is given by (2) in terms of the classical double r -matrix, $r = \sum_i E^i \otimes J_i$. We can then deduce the Poisson structure for the variables u and ℓ ,

$$\{\ell_1, \ell_2\} = -[r, \ell_1 \ell_2], \quad \{\ell_1, u_2\} = -\ell_1 r u_2, \quad \{u_1, \ell_2\} = \ell_2 r^t u_1, \quad \{u_1, u_2\} = -[r^t, u_1 u_2]. \quad (6)$$

When using the explicit parametrization of u and ℓ , these Poisson brackets become the standard $T^*\text{SU}(2)$ Poisson structure:

$$\{X_i, X_j\} = \epsilon_{ij}^k X_k, \quad \{X_i, \mathbf{R}\} = -J_i \mathbf{R}, \quad \{\mathbf{R}, \mathbf{R}\} = 0, \quad (7)$$

Hence $\text{ISO}(3)$ equipped with the Poisson structure (6) is an equivalent description of the phase space $T^*\text{SU}(2)$, which is commonly used in LQG.

Instead of considering the left decomposition of $\text{ISO}(3)$, $D = \ell u$, we could consider the right decomposition $D = \tilde{u} \tilde{\ell}$. Since, $\tilde{u} \tilde{\ell} = \ell u$ we have that

$$D = (\mathbf{R}, \mathbf{X}) = \tilde{u} \tilde{\ell}, \quad \text{with } \tilde{\ell} = (\mathbf{1}, \tilde{\mathbf{X}}) = (\mathbf{1}, \mathbf{R}^{-1} \triangleright \mathbf{X}), \quad \tilde{u} = (\mathbf{R}, \vec{0}). \quad (8)$$

So $\tilde{u} = u$ and $\tilde{\ell} = u^{-1} \ell u$.

This factorization comes from building the classical double of the Abelian Lie algebra \mathbb{R}^3 equipped with the non trivial cocycle $\delta_*(E^i) = \epsilon_{ij}^k E^i \otimes E^j$. The dual bialgebra is then $\mathfrak{so}(3)$ with a trivial cocycle. The relevant r -matrix for the classical double $\mathfrak{d}_0(\mathbb{R}^3)$ is then $\tilde{r} = \sum_i J^i \otimes E_i = r^t$. The symplectic structure on $\text{ISO}(3) = \text{SO}(3) \ltimes \mathbb{R}^3$ is given by (3) which implies the following Poisson brackets for the \tilde{u} and $\tilde{\ell}$ variables,

$$\{\tilde{\ell}_1, \tilde{\ell}_2\} = -[r, \tilde{\ell}_1 \tilde{\ell}_2], \quad \{\tilde{\ell}_1, \tilde{u}_2\} = -\tilde{\ell}_1 r \tilde{u}_2, \quad \{\tilde{u}_1, \tilde{\ell}_2\} = \tilde{\ell}_2 r^t \tilde{u}_1, \quad \{\tilde{u}_1, \tilde{u}_2\} = -[r^t, \tilde{u}_1 \tilde{u}_2]. \quad (9)$$

In terms of the $T^*\text{SU}(2)$ variables, we have then

$$\{\tilde{X}_i, \tilde{X}_j\} = \epsilon_{ij}^k \tilde{X}_k, \quad \{\tilde{X}_i, \mathbf{R}\} = \mathbf{R} J_i, \quad \{\mathbf{R}, \mathbf{R}\} = 0. \quad (10)$$

We see that the two decompositions of the Heisenberg double built on $\text{ISO}(3)$ correspond to considering the left and right invariant vectors on $\text{SO}(3)$.

An explicit realization of the generators of $\mathfrak{so}(3)$ and \mathbb{R}^3 as well as of the r -matrices, r and r^t , and of the variables ℓ and u can be found in Appendix A.

The explicit link between the variables of the left and of the right decompositions of $D \in \text{ISO}(3)$ allows us to evaluate the Poisson brackets between $\tilde{\ell}$ and ℓ and u and between \tilde{u} and ℓ and u . Indeed, $\tilde{u} = u$ implies that

$$\{\tilde{u}_1, u_2\} = 0, \quad \{\tilde{u}_1, \ell_2\} = \ell_2 r^t \tilde{u}_1, \quad (11)$$

and $\tilde{\ell} = u^{-1} \ell u$ implies that

$$\{\tilde{\ell}_1, \ell_2\} = 0, \quad \{\tilde{\ell}_1, u_2\} = u_2 r \tilde{\ell}_1. \quad (12)$$

C. $\text{SL}(2, \mathbb{C})$ as a phase space

We construct the Heisenberg double structure on $\text{SL}(2, \mathbb{C})$, following the same recipe as above. The factorization of $\text{SL}(2, \mathbb{C})$ into $\text{SU}(2)$ and $\text{SB}(2, \mathbb{C})$ is given by the Iwasawa decomposition. As in the Euclidian group case, we have a left and a right decomposition. We construct the phase space structure for each if these decompositions. We are going to use the spinorial representation following the standard literature. In particular, we use the spinorial representation of $\mathfrak{su}(2)$, so we use as generators, the Pauli matrices with commutation relations $[\sigma_i, \sigma_j] = 2i \epsilon_{ij}^k \sigma_k$ (see appendix B for the expression of the Pauli matrices).

1. Left Iwasawa decomposition

Our starting point is now $\mathfrak{su}(2)$ equipped with a non trivial cocycle.

$$\delta(\sigma_k) = 2i \kappa \left(\delta_k^i \delta_z^j - \delta_k^j \delta_z^i \right) \sigma_i \otimes \sigma_j. \quad (13)$$

The classical double $\mathfrak{d}(\mathfrak{su}(2))$ is then given by the following structures.

- We note τ^i the dual of σ_i . The \mathfrak{d} -invariant bilinear form is $\langle \tau^i, \sigma_j \rangle = \delta_j^i$, $\langle \sigma_i, \sigma_j \rangle = \langle \tau^i, \tau^j \rangle = 0$.
- The dual Lie algebra $\mathfrak{su}(2)^*$ is not Abelian anymore since the $\mathfrak{su}(2)$ cocycle δ in (13) is non trivial. We have

$$[\tau^i, \tau^j] = 2i \kappa (\delta_{jz} \tau^i - \delta_{iz} \tau^j). \quad (14)$$

Therefore, we have that $\mathfrak{su}(2)^* = \mathfrak{sb}(2, \mathbb{C})$, the Lie algebra which generates the 2×2 lower triangular matrices. The generators τ^i can be expressed in terms of the Pauli matrices.

$$\tau^i = i \kappa (\sigma_i - \frac{1}{2} [\sigma_z, \sigma_i]) = \kappa (i \sigma_i + \epsilon_{zi}^k \sigma_k). \quad (15)$$

Their explicit expression as 2-by-2 matrices can be found in appendix B. The $\mathfrak{sb}(2, \mathbb{C})$ cocycle is induced by the Lie algebra structure of $\mathfrak{su}(2)$, so $\delta_*(\tau^k) = 2i \epsilon_{ij}^k \tau^i \otimes \tau^j$.

- As a Lie algebra $\mathfrak{d} = \mathfrak{sb}(2, \mathbb{C}) \ltimes \mathfrak{su}(2) \sim \mathfrak{sl}(2, \mathbb{C})$.
- The (coboundary) cocycle $\delta_{\mathfrak{d}}$ is generated by the r -matrix

$$r = \frac{1}{4} \sum_i \tau^i \otimes \sigma_i = \frac{i \kappa}{4} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & 4 & -1 & \\ & & & 1 \end{pmatrix}. \quad (16)$$

We have constructed above the classical double $\mathfrak{d}(\mathfrak{su}(2))$, which leads to the Lie algebra $\mathfrak{sb}(2, \mathbb{C}) \bowtie \mathfrak{su}(2) \sim \mathfrak{sl}(2, \mathbb{C})$. This corresponds to the "left Iwasawa decomposition" of $\mathfrak{sl}(2, \mathbb{C})$. This decomposition is naturally extended to the group element $D \in \text{SL}(2, \mathbb{C})$.

$$D = \ell u, \quad u = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in \text{SU}(2), \quad \ell = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix} \in \text{SB}(2, \mathbb{C}), \quad \lambda \in \mathbb{R}_+^*, \quad z \in \mathbb{C}. \quad (17)$$

It is common in the literature to express the Poisson structure in terms of the Hermitian conjugated r^\dagger instead of the transpose r^t [19]. Note that in this particular case, $r^\dagger = \sum_i (\tau^i)^\dagger \otimes \sigma_i = -\sum_i \sigma_i \otimes \tau^i = -r^t$. Hence the symplectic structure between the matrix elements of $\text{SL}(2, \mathbb{C})$ becomes

$$\{D_1, D_2\} = -r D_1 D_2 - D_1 D_2 r^\dagger, \quad D \in \text{SL}(2, \mathbb{C}). \quad (18)$$

The full phase space structure for the configuration variables u and momenta ℓ now is given by

$$\{\ell_1, \ell_2\} = -[r, \ell_1 \ell_2], \quad \{\ell_1, u_2\} = -\ell_1 r u_2, \quad \{u_1, \ell_2\} = -\ell_2 r^\dagger u_1, \quad \{u_1, u_2\} = [r^\dagger, u_1 u_2]. \quad (19)$$

Note that since z is complex, we also need to consider its conjugated \bar{z} . To this aim, we introduce $l = (\ell^\dagger)^{-1}$,

$$l = \begin{pmatrix} \lambda^{-1} & -\bar{z} \\ 0 & \lambda \end{pmatrix}. \quad (20)$$

To specify, the Poisson brackets of l with ℓ and u , we use the fact that the $\text{SB}(2, \mathbb{C})$ group and the algebra $\mathfrak{sb}(2, \mathbb{C})$ are preserved under $\ell \rightarrow l, \tau^i \rightarrow (\tau^i)^\dagger$. As in [19], we require that the Poisson brackets of u and ℓ are preserved under this mapping as well.

$$\begin{aligned} \{l_1, \ell_2\} &= -\frac{1}{4}[(\tau^i)^\dagger \otimes \sigma_i, l_1 \ell_2] = -[r^\dagger, l_1 \ell_2], \quad \{l_1, u_2\} = -\frac{1}{4}l_1((\tau^i)^\dagger \otimes \sigma_i)u_2 = -l_1 r^\dagger u_2, \\ \{l_1, l_2\} &= -\frac{1}{4}[(\tau^i)^\dagger \otimes \sigma_i, l_1 l_2] = -[r^\dagger, l_1 l_2]. \end{aligned} \quad (21)$$

The explicit expression of these Poisson brackets in terms of the matrix elements λ, z, \bar{z} and α, β is given in Appendix B.

The Heisenberg double $\text{SL}(2, \mathbb{C})$ can be seen as a deformation of the Heisenberg double $\text{ISO}(3)$. We shall discuss the limit in section ID. In this latter case, angular momentum variables are given by $\mathbf{X} \in \mathbb{R}^3$. When moving to the $\text{SL}(2, \mathbb{C})$ case, we have deformed \mathbb{R}^3 into the hyperboloid of radius κ^{-1} . Since we are using the spinorial representation, a point on the hyperboloid can be obtained from the combinations DD^\dagger and $D^\dagger D$. We construct the two vector-like quantities out of these two combinations,

$$DD^\dagger \equiv L = \ell \ell^\dagger = \begin{pmatrix} \lambda^2 & \lambda \bar{z} \\ \lambda z & \lambda^{-2} + |z|^2 \end{pmatrix}, \quad T_0 = \frac{1}{2\kappa} \text{Tr} L = \frac{1}{2\kappa} (\lambda^2 + \lambda^{-2} + |z|^2), \quad \mathbf{T} \equiv \frac{1}{2\kappa} \text{Tr}(\ell \ell^\dagger \vec{\sigma}), \quad (22)$$

with explicit components

$$T_z = \frac{1}{2\kappa} \text{Tr} \ell \ell^\dagger \sigma_z = \frac{1}{2\kappa} (\lambda^2 - \lambda^{-2} - |z|^2), \quad T_+ = \frac{1}{\kappa} \text{Tr} \ell \ell^\dagger \sigma_+ = \frac{1}{\kappa} \lambda z, \quad T_- = \overline{T_+} = \frac{1}{\kappa} \lambda \bar{z}. \quad (23)$$

The 4-vector T^μ defines indeed a point on the hyperboloid since we have $T^2 = T_\mu T^\mu = T_0^2 - \vec{T}^2 = T_0^2 - (T_z^2 + T_+ T_-) = 1/\kappa^2$. Using either the formula with the r -matrix or the explicit commutators of z, \bar{z} and λ , we can compute the Poisson brackets between the T_μ components.

$$\{T_0, \mathbf{T}\} = 0, \quad \{T_z, T_+\} = i\kappa (T_0 + T_z) T_+, \quad \{T_z, T_-\} = -i\kappa (T_0 + T_z) T_-, \quad \{T_+, T_-\} = 2i\kappa (T_0 + T_z) T_z, \quad (24)$$

where $\kappa(T_0 + T_z) = \lambda^2$ plays a role of a re-scaling compared to the commutators of the standard $\mathfrak{su}(2)$ algebra.

The other vector we can construct is the analogue of the vector $\tilde{\mathbf{X}} = \mathbf{R}^{-1} \triangleright \mathbf{X}$.

$$uD^\dagger Du^{-1} \equiv L^{op} \equiv \ell^\dagger \ell = \begin{pmatrix} \lambda^2 + |z|^2 & \lambda^{-1} \bar{z} \\ \lambda^{-1} z & \lambda^{-2} \end{pmatrix}, \quad T_0^{op} = T_0, \quad \mathbf{T}^{op} \equiv \frac{1}{2\kappa} \text{Tr}(\ell^\dagger \ell \vec{\sigma}), \quad (25)$$

with explicit expressions:

$$T_z^{op} = \frac{1}{2\kappa} (\lambda^2 - \lambda^{-2} + |z|^2), \quad T_+^{op} = \frac{1}{\kappa} \lambda^{-1} z, \quad T_-^{op} = \frac{1}{\kappa} \lambda^{-1} \bar{z}. \quad (26)$$

2. Right Iwasawa decomposition

Instead of building the classical double from $\mathfrak{su}(2)$, we could have constructed it out of $\mathfrak{sb}(2, \mathbb{C})$, with the non trivial cocycle $\delta_*(\tau^k) = 2i\epsilon_{ij}^k \tau^i \otimes \tau^j$. This leads to the "right Iwasawa decomposition" of $\mathfrak{sl}(2, \mathbb{C}) \sim \mathfrak{su}(2) \ltimes \mathfrak{sb}(2, \mathbb{C}) \sim \mathfrak{d}_{\tilde{r}}(\mathfrak{sb}(2, \mathbb{C}))$. The associated r -matrix is then $\tilde{r} = \frac{1}{4} \sum \sigma_i \otimes \tau_i = r^t = -r^\dagger$. This decomposition can be extended to group elements.

$$D = \tilde{u}\tilde{\ell}, \quad \tilde{u} = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta} \\ \tilde{\beta} & \tilde{\alpha} \end{pmatrix} \in \text{SU}(2), \quad \tilde{\ell} = \begin{pmatrix} \tilde{\lambda} & 0 \\ \tilde{z} & \tilde{\lambda}^{-1} \end{pmatrix} \in \text{SB}(2, \mathbb{C}), \quad \tilde{\lambda} \in \mathbb{R}_+^*, \quad \tilde{z} \in \mathbb{C}. \quad (27)$$

The phase space structure is then given by \tilde{r} .

$$\{D_1, D_2\} = -\tilde{r}D_1D_2 - D_1D_2\tilde{r}^\dagger = r^\dagger D_1D_2 + D_1D_2r.$$

The Poisson structure on the variables $\tilde{u}, \tilde{\ell}$ and \tilde{l} , where \tilde{l} is defined as previously by $\tilde{l} = (\tilde{\ell}^\dagger)^{-1}$ is the following,

$$\begin{aligned} \{\tilde{\ell}_1, \tilde{\ell}_2\} &= -[r, \tilde{\ell}_1\tilde{\ell}_2], & \{\tilde{\ell}_1, \tilde{u}_2\} &= \tilde{u}_2r\tilde{\ell}_1, & \{\tilde{u}_1, \tilde{u}_2\} &= [r^\dagger, \tilde{u}_1\tilde{u}_2], \\ \{\tilde{l}_1, \tilde{\ell}_2\} &= -[r^\dagger, \tilde{l}_1\tilde{\ell}_2], & \{\tilde{l}_1, \tilde{l}_2\} &= -[r^\dagger, \tilde{l}_1\tilde{l}_2], & \{\tilde{u}_1, \tilde{\ell}_2\} &= \tilde{u}_1r^\dagger\tilde{\ell}_2. \end{aligned} \quad (28)$$

As earlier, we can introduce the different vectors associated to this decomposition.

$$\tilde{L} = \tilde{\ell}\tilde{\ell}^\dagger = \begin{pmatrix} \tilde{\lambda}^2 & \tilde{\lambda}\tilde{z} \\ \tilde{\lambda}\tilde{z} & \tilde{\lambda}^{-2} + |\tilde{z}|^2 \end{pmatrix}, \quad \tilde{\mathbf{T}} = \frac{1}{2\kappa} \text{Tr}(\tilde{\ell}\tilde{\ell}^\dagger\tilde{\sigma}), \quad \tilde{T}_z = \frac{1}{2\kappa} (\tilde{\lambda}^2 - \tilde{\lambda}^{-2} - |\tilde{z}|^2), \quad \tilde{T}_+ = \frac{1}{\kappa} \tilde{\lambda}\tilde{z}, \quad \tilde{T}_- = \frac{1}{\kappa} \tilde{\lambda}\tilde{z}^*, \quad (29)$$

Note that the two decompositions describe the same group element $D = \ell u = \tilde{u}\tilde{\ell}$. Hence we have that $\tilde{L} = \tilde{u}^{-1}\ell\ell^\dagger\tilde{u} = \tilde{u}^{-1}L\tilde{u}$. As a consequence, the 3-vector $\tilde{\mathbf{T}}$ is related to the 3-vector \mathbf{T} by the rotation \tilde{u}^{-1} , $\tilde{\mathbf{T}} = \tilde{u}^{-1} \triangleright \mathbf{T}$.

We can also construct the other vector $\tilde{\mathbf{T}}^{op}$, given by the other combination $\tilde{\ell}^\dagger\tilde{\ell}$.

$$\tilde{\mathbf{T}}^{op} \equiv \frac{1}{2\kappa} \text{Tr}(\tilde{\ell}^\dagger\tilde{\ell}\tilde{\sigma}), \quad \tilde{T}_z^{op} = \frac{1}{2\kappa} (\tilde{\lambda}^2 - \tilde{\lambda}^{-2} + |\tilde{z}|^2), \quad \tilde{T}_+^{op} = \frac{1}{\kappa} \tilde{\lambda}^{-1}\tilde{z}, \quad \tilde{T}_-^{op} = \frac{1}{\kappa} \tilde{\lambda}^{-1}\tilde{z}^*. \quad (30)$$

Since $\tilde{L}^{op} = \tilde{\ell}^\dagger\tilde{\ell} = u^{-1}\ell^\dagger\ell u = u^{-1}L^{op}u$, the 3-vector $\tilde{\mathbf{T}}^{op}$ is related to the 3-vector \mathbf{T}^{op} by the rotation u^{-1} , $\tilde{\mathbf{T}}^{op} = u^{-1} \triangleright \mathbf{T}^{op}$.

This explicit link between the variable of the left and right Iwasawa decomposition of $D \in \text{SL}(2, \mathbb{C})$ allows us to express the Poisson brackets between the ℓ, l, u variables and the $\tilde{\ell}, \tilde{l}, \tilde{u}$ variables. By considering $\tilde{\ell}^\dagger\tilde{\ell} = u^{-1}\ell^\dagger\ell u$, we deduce that

$$\{\tilde{\ell}_1^{-1}, u_2\} = u_2 r \tilde{\ell}_1^{-1}, \quad \{\tilde{\ell}_1, \ell_2\} = 0. \quad (31)$$

From the first equality of the above equation, we can in particular infer that

$$\{\tilde{l}_1, u_2\} = -\tilde{l}_1 u_2 r^\dagger. \quad (32)$$

Finally, using $\tilde{u} = \ell u \tilde{\ell}^{-1}$, we get

$$\{\tilde{u}_1, u_2\} = 0, \quad \{\tilde{u}_1, \ell_2\} = -r^\dagger \tilde{u}_1 \ell_2. \quad (33)$$

More details about the computations of these commutation relations and the commutation relations in terms of the components of $\tilde{\ell}, \tilde{l}, \ell, u$ and \tilde{u} can be found in the Appendix B.

D. $\text{SL}(2, \mathbb{C})$ phase space as a deformation of the $\text{ISO}(3)$ phase space

We would like to show here how the $\text{ISO}(3)$ phase space can be obtained in the limit $\kappa \rightarrow 0$, so that $\text{SL}(2, \mathbb{C})$ as a phase space can be viewed as a deformation of the $\text{ISO}(3)$ phase space.

As groups, we know that the Ionu-Wigner contraction allows to recover $\text{ISO}(3)$ from $\text{SL}(2, \mathbb{C})$. This contraction is implemented through $\kappa \rightarrow 0$, such that $\tau_i \rightarrow E_i$, the generators of the translations. Indeed the $\mathfrak{sb}(2, \mathbb{C})$ Lie bracket becomes the \mathbb{R}^3 Lie brackets.

$$[\tau_i, \tau_j] = 2i\kappa(\delta_{iz}\tau_j - \delta_{jz}\tau_i) \rightarrow 0 = [E_i, E_j]. \quad (34)$$

Due to the difference of representation (spinor for $\text{SL}(2, \mathbb{C})$ versus vector for $\text{ISO}(3)$), we might wonder what the relations between the different coordinates used to parameterize the two momentum spaces \mathbb{R}^3 and $\text{SB}(2, \mathbb{C})$ are. By construction, the generators τ_i have dimension κ , so that the coordinates on $\text{SB}(2, \mathbb{C})$ should have dimension κ^{-1} , in agreement with the geometric interpretation of the momentum space as the hyperboloid of radius κ^{-1} . If we consider the "Euler" parametrization of $\ell \in \text{SB}(2, \mathbb{C})$, we consider $\mathbf{j} \in \mathbb{R}^3$ with dimension κ^{-1} such that

$$\ell = e^{ij_z \tau^z} e^{ij_x \tau^x} e^{ij_y \tau^y}, \quad \lambda = e^{-\frac{\kappa}{2} j_z}, \quad z = -\kappa e^{\frac{\kappa}{2} j_z} (j_x + ij_y) = -e^{\frac{\kappa}{2} j_z} \kappa j_+, \quad \bar{z} = -e^{\frac{\kappa}{2} j_z} \kappa j_-. \quad (35)$$

We would like to check that the Poisson brackets of the \mathbf{j} 's expressed in terms of λ, z, \bar{z} give rise to the Poisson brackets between the usual angular momentum variables \mathbf{X} in the limit $\kappa \rightarrow 0$. We get

$$\{j_z, j_+\} = \frac{i}{\kappa} \lambda z = -ij_+, \quad \{j_z, j_-\} = \frac{i}{\kappa} \lambda \bar{z} = ij_-, \quad \{j_+, j_-\} = \frac{i}{\kappa} (\kappa^2 j_+ j_- + e^{-2\kappa j_z} - 1) \xrightarrow{\kappa \rightarrow 0} -2ij_z. \quad (36)$$

Hence we have recovered the standard angular momentum Poisson relations, modulo the fact that j_+ and j_- are swapped with respect to the usual commutation relations, i.e. $j_+ \rightarrow X_-$, $j_- \rightarrow X_+$ and $j_z \rightarrow X_z$.

For the sake of completeness, we can also take the limit $\kappa \rightarrow 0$ of the vector \mathbf{T} in terms of \mathbf{j} ,

$$T_+ = -j_+, \quad T_- = -j_-, \quad T_z = \frac{1}{2\kappa} (e^{-\kappa j_z} - e^{\kappa j_z} + \kappa^2 e^{-\kappa j_z} j_+ j_-) \xrightarrow{\kappa \rightarrow 0} -j_z. \quad (37)$$

The Poisson brackets (24) between the components of \mathbf{T} also reproduce the angular momentum Poisson algebra,

$$\kappa T_0 \rightarrow 1, \quad \{T_z, T_\pm\} \rightarrow \pm iT_\pm, \quad \{T_+, T_-\} \rightarrow 2iT_z, \quad (38)$$

meaning that \mathbf{T} and \mathbf{X} coincide in the limit $\kappa \rightarrow 0$.

The bottom line is that $\text{SL}(2, \mathbb{C})$ as a phase space can be viewed as a deformation of $\text{ISO}(3)$ seen as a phase space.

II. POISSON LIE SYMMETRIES

The introduction of Poisson Lie groups is motivated by the identification of the phase space symmetries. They must be compatible with the Poisson structure, meaning that the symmetry action is a *Poisson map* [16]. The symmetries are then called *Poisson-Lie group symmetries*. Having such a symmetry amounts to putting a Poisson structure² on the symmetry group, which is compatible with the group product (i.e. the group multiplication is a Poisson map). In this case we have a Poisson Lie group as symmetry group. Often this Poisson structure does not need to be specified, but when the phase space symplectic structure is non trivial, the Poisson structure on the symmetry group also becomes non trivial and cannot be overlooked.

In the previous section, we have considered some examples of Poisson Lie groups. We had for $G = \text{SU}(2)$ or $\text{SO}(3)$ and its dual G^* ,

$$(G, \{\cdot, \cdot\}_G) \rightsquigarrow \{u_1, u_2\}_G = -[r^t, u_1 u_2], \quad (G^*, \{\cdot, \cdot\}_{G^*}) \rightsquigarrow \{\ell_1, \ell_2\}_G = -[r, \ell_1 \ell_2]. \quad (39)$$

From G and G^* , we constructed $\mathcal{D} = G^* \bowtie G$. As a phase space, \mathcal{D} was equipped with the symplectic structure π_+ of the Heisenberg double which is not a Poisson Lie group structure. \mathcal{D} can also be equipped with the Poisson structure

$$\pi_-(\mathcal{D}) = -[\mathbf{a}, D_1 \otimes D_2] = -[r, D_1 D_2], \quad D \in \mathcal{D}. \quad (40)$$

Then, $\mathcal{D} = \mathcal{D}_{\pi_-}$ is a Poisson Lie group and is called a *Drinfeld double*. The Drinfeld double \mathcal{D}_{π_-} can actually be seen as the symmetry structure behind the Heisenberg double \mathcal{D}_{π_+} as the following theorem shows.

Theorem II.1. *Drinfeld double as symmetry structure*

The multiplication maps

$$\begin{array}{ccc} \mathcal{D}_{\pi_-} \times \mathcal{D}_{\pi_+} & \rightarrow & \mathcal{D}_{\pi_+} \\ (U, D) & \rightarrow & UD \end{array} \quad \begin{array}{ccc} \mathcal{D}_{\pi_+} \times \mathcal{D}_{\pi_-} & \rightarrow & \mathcal{D}_{\pi_+} \\ (D, V) & \rightarrow & DV \end{array}$$

² It will never be symplectic by construction.

are Poisson maps, so they define left and right Poisson actions of respectively \mathcal{D}_{π_-} and $\mathcal{D}_{-\pi_-}$ on \mathcal{D}_{π_+} , [18]. Explicitly, we have the symmetry transformations U and V that do Poisson commute with D an element of the Heisenberg double and

$$\{U_1 D_1, U_2 D_2\}_{\pi_+} = \{U_1, U_2\}_{\pi_-} D_1 D_2 + U_1 U_2 \{D_1, D_2\}_{\pi_+}, \quad (41)$$

$$\{D_1 V_1, D_2 V_2\}_{\pi_+} = \{D_1, D_2\}_{\pi_+} V_1 V_2 - D_1 D_2 \{V_1, V_2\}_{\pi_-}, \quad U, V, D \in \mathcal{D}. \quad (42)$$

Therefore, \mathcal{D}_{π_-} equipped with (40) is the symmetry structure behind a phase space with symplectic structure defined by π_+ . The symmetries of the phase spaces considered in the previous section, namely $\mathcal{D}_{\pi_+} = \text{ISO}(3)$ and $\text{SL}(2, \mathbb{C})$, are respectively $\mathcal{D}_{\pi_-} = \text{ISO}(3)$, $\text{SL}(2, \mathbb{C})$ equipped with (40) which makes them Poisson Lie groups. The r -matrix of (40) is the one in the definition of the symplectic structure of \mathcal{D}_{π_+} given by (A4) for $\text{ISO}(3)$ and (16) for $\text{SL}(2, \mathbb{C})$. Explicitly, if we write an element D seen as generating a symmetry transformation in the left decomposition $D = mv$, with $m \in G^*$ and $v \in G$, the different Poisson brackets read

$$\{m_1, m_2\}_{\pi_-} = -[r, m_1 m_2], \quad \{v_1, v_2\}_{\pi_-} = -[r^t, v_1 v_2], \quad \{m_1, v_2\}_{\pi_-} = \{v_1, m_2\}_{\pi_-} = 0. \quad (43)$$

If $G = \text{SU}(2)$ or $\text{SO}(3)$, v is a rotation and m is a translation in the case $G^* = \mathbb{R}^3$ or a deformed translation for $G^* = \text{SB}(2, \mathbb{C})$. From the previous commutation relations, we can see that in general the coordinates encoding the symmetry transformation will not Poisson commute. Upon quantization, they will then give rise to the notion of *quantum group*. Indeed Poisson Lie groups are the classical version of a quantum group [17].

To have the explicit action of the rotations v and the (deformed) translations m on the phase space variables ℓ and u , we split the action of the Drinfeld double \mathcal{D}_{π_-} on the Heisenberg double \mathcal{D}_{π_+} into the action of its subgroups, G and G^* on \mathcal{D}_{π_+} . The left or right actions of G and G^* by multiplication on the Heisenberg double \mathcal{D}_{π_+} are given below where we have chosen the left decomposition of $D = \ell u \in \mathcal{D}_{\pi_+}$. The left and right (deformed) translations are given by the action of G^* on \mathcal{D} .

$$D \rightarrow m D = m \ell u = {}^{(m)}\ell {}^{(m)}u \rightarrow \begin{cases} {}^{(m)}u = u \\ {}^{(m)}\ell = m\ell \end{cases}, \quad D \rightarrow D m = \ell u m = \ell {}^{(m)}u \rightarrow \begin{cases} {}^{(u)}m u {}^{(m)} = m u \\ \ell {}^{(m)} = \ell ({}^{(u)}m) \end{cases}. \quad (44)$$

Note that to keep track on the side of the action, we use the (standard) notation $\ell^{(m)}$ and $u^{(m)}$ for the right action of m , as opposed to ${}^{(m)}\ell$ and ${}^{(m)}u$ for the left action. We use a similar notation for the rotation v . The left and right rotations are given by the action of G on \mathcal{D} .

$$D \rightarrow v D = v \ell u = {}^{(v)}\ell {}^{(v)}u \rightarrow \begin{cases} {}^{(v)}u = v {}^{(\ell)}u \\ {}^{(v)}\ell v {}^{(\ell)} = v\ell \end{cases}, \quad D \rightarrow D v = \ell u v = \ell {}^{(v)}u \rightarrow \begin{cases} u {}^{(v)} = u v \\ \ell {}^{(v)} = \ell \end{cases}. \quad (45)$$

In the case $\mathcal{D} = \text{ISO}(3)$, these transformations take a well known form. Indeed, if we denote $m = (\mathbf{1}, \mathbf{a})$, $v = (g, \mathbf{0})$, $\ell = (\mathbf{1}, \mathbf{X})$, $u = (\mathbf{R}, \mathbf{0})$, we recover that

$$\begin{aligned} \text{left translations: } \begin{cases} {}^{(m)}u = u \\ {}^{(m)}\ell = m\ell \rightsquigarrow \mathbf{X} \rightarrow \mathbf{X} + \mathbf{a} \end{cases} & \quad \text{right translations: } \begin{cases} {}^{(u)}m u {}^{(m)} = u m \rightsquigarrow \begin{cases} {}^{(u)}m = u m u^{-1}, \\ u {}^{(m)} = u \end{cases} \\ \ell {}^{(m)} = \ell ({}^{(u)}m) \rightsquigarrow \mathbf{X} \rightarrow \mathbf{X} + \mathbf{R} \triangleright \mathbf{a} \end{cases} \\ \text{left rotations: } \begin{cases} {}^{(v)}u = v u, {}^{(\ell)}v = v, \\ {}^{(v)}\ell = v \ell v^{-1} \rightsquigarrow \mathbf{X} \rightarrow g \triangleright \mathbf{X} \end{cases} & \quad \text{right rotations: } \begin{cases} u {}^{(v)} = u v \\ \ell {}^{(v)} = \ell \end{cases} \end{aligned} \quad (46)$$

The interpretation of the $\text{SL}(2, \mathbb{C})$ case is more complicated but some analogues of the above equations can be found in the appendix C.

In order to relate the constraints introduced in the next section to the above Poisson Lie symmetries, we express the infinitesimal version of the Poisson Lie symmetries (44) and (45) in terms of Poisson brackets on the *Heisenberg double*. Indeed, later on we will build a lattice gauge theory with constraints which are functions over several copies of the $\text{SL}(2, \mathbb{C})$ phase space. We will have to show that the Poisson brackets from the symplectic structure $\{\cdot, \cdot\}_{\pi_+}$ with the constraints generate gauge symmetries. Due to the different representations we have been using in the $\text{ISO}(3)$ and $\text{SL}(2, \mathbb{C})$ cases (vector versus spinor representations), the formulae will not look exactly the same.

The proofs of the following propositions consists on one hand in determining the infinitesimal version of (44) and (45) and on the other hand calculating explicitly the Poisson brackets we will give to get a match. Most of these details are found in the appendix C.

In the following, we consider $m \sim \mathbf{1} + \delta m$ and $v \sim \mathbf{1} + i\vec{\varepsilon} \cdot \vec{J}$, where \vec{J} are the $\mathfrak{su}(2)$ generators in the relevant representation.

Proposition II.2. Translations for ISO(3)

For $\mathcal{D}_+ = \text{ISO}(3)$, we consider $\delta m = \vec{\varepsilon} \cdot \vec{E}$ and we recall that we have the duality $\langle E_i; J_j \rangle = \delta_{ij}$. The infinitesimal right translation (44) acting on ℓ and u can then be rewritten as

$$\delta_R^{(m)} u = \langle \delta m_1; u_1^{-1} \{u_1, u_2\} \rangle_1 = 0, \quad \delta_R^{(m)} \ell = \langle \delta m_1; u_1^{-1} \{u_1, \ell_2\} \rangle_1 = \ell u \delta m u^{-1} \quad (47)$$

The infinitesimal left translation acting on ℓ and u can be rewritten as

$$\delta_L^{(m)} u = \langle \delta m_1; \{\tilde{u}_1, u_2\} \tilde{u}_1^{-1} \rangle_1 = 0 \quad \delta_L^{(m)} \ell = \langle \delta m_1; \{\tilde{u}_1, \ell_2\} \tilde{u}_1^{-1} \rangle_1 = \delta m \ell \quad (48)$$

In the $\text{SB}(2, \mathbb{C})$ case, δm cannot not contain the full information, as we have seen in the previous section when we used ℓ and ℓ^\dagger to build a vector from $L = \ell \ell^\dagger$. So we need to consider δm^\dagger as well and to this goal we introduce $M = m m^\dagger$ and $\delta M = \delta m + \delta m^\dagger$. The infinitesimal deformed translations for the $\text{SL}(2, \mathbb{C})$ case are given by the following proposition.

Proposition II.3. Deformed translations for $\text{SL}(2, \mathbb{C})$

We use Tr_1 for the trace on the first factor of the tensor product. The infinitesimal right and left translation acting on ℓ and u can be rewritten as

$$\begin{aligned} \delta_R^{(m)} u &= -i\kappa^{-1} \text{Tr}_1 \delta M_1 u_1^{-1} \{u_1, u_2\} = \frac{1}{4} u [\delta M, \sigma_z - u^{-1} \sigma_z u], \\ \delta_R^{(m)} \ell &= -i\kappa^{-1} \text{Tr}_1 \delta M_1 u_1^{-1} \{u_1, \ell_2\} = \frac{1}{2} \ell \left(u \delta M u^{-1} + \frac{1}{2} [u \delta M u^{-1}, \sigma_z] \right). \end{aligned} \quad (49)$$

The infinitesimal left translation acting on ℓ and u can be rewritten as

$$\delta_L^{(m)} u = -\frac{i}{4\kappa} \text{Tr}_1 \delta M_1 \{\tilde{u}_1, u_2\} \tilde{u}_1^{-1} = -\frac{i}{4\kappa} \text{Tr}_1 \delta M_1 0 = 0, \quad (50)$$

$$\delta_L^{(m)} \ell = -\frac{i}{4\kappa} \text{Tr}_1 \delta M_1 \{\tilde{u}_1, \ell_2\} \tilde{u}_1^{-1} = \frac{i}{4\kappa} \sum_i \text{Tr}(\delta M \tau_i^\dagger) \sigma_i \ell = \delta m \ell, \quad (51)$$

We now address the rotations. As expected, they are generated by the variables ℓ and $\tilde{\ell}$ in the $\text{ISO}(3)$ case.

Proposition II.4. Rotations for ISO(3)

For $\mathcal{D}_+ = \text{ISO}(3)$, consider a $\text{SU}(2)$ group element $v \sim \mathbf{1} + i\vec{\varepsilon} \cdot \vec{J} = \mathbf{1} + V$ where \vec{J} is in the vector representation. Then the variations of ℓ and u under a left infinitesimal $\text{SU}(2)$ transformations are given by

$$\delta_L^v \ell = -\langle V_1; \ell_1^{-1} \{\ell_1, \ell_2\} \rangle_1 = V \ell - \ell V \quad \delta_L^v u = -\langle V_1; \ell_1^{-1} \{\ell_1, u_2\} \rangle_1 = V u \quad (52)$$

The variations of ℓ and u under a right infinitesimal $\text{SU}(2)$ transformations are given instead by the Poisson brackets with the Hermitian matrix $\tilde{\ell}$.

$$\delta_R^v \ell = \langle V_1; \tilde{\ell}_1^{-1} \{\tilde{\ell}_1, \ell_2\} \rangle_1 = 0, \quad \delta_R^v u = \langle V_1; \tilde{\ell}_1^{-1} \{\tilde{\ell}_1, u_2\} \rangle_1 = u V \quad (53)$$

In the $\text{SB}(2, \mathbb{C})$ case, ℓ does not contain the full information about the flux, we need to consider also ℓ^\dagger . Hence, we can use $L = \ell \ell^\dagger$ and $\tilde{L}^{op} = \tilde{\ell}^\dagger \tilde{\ell}$ to generate the left and right rotations.

Proposition II.5. Rotations for $\text{SL}(2, \mathbb{C})$

For $\mathcal{D}_+ = \text{SL}(2, \mathbb{C})$, consider a $\text{SU}(2)$ group element $v = \mathbf{1} + i\vec{\varepsilon} \cdot \vec{\sigma}$ and the matrix V

$$v = \mathbf{1} + i(V - \frac{1}{2} \text{Tr } V \mathbf{1}), \quad V = \begin{pmatrix} 2\epsilon_z & \epsilon_- \\ \epsilon_+ & 0 \end{pmatrix} = \epsilon_z(\mathbf{1} + \sigma_z) + \epsilon_- \sigma_+ + \epsilon_+ \sigma_- . \quad (54)$$

Then the variations of ℓ and u under a left infinitesimal $\text{SU}(2)$ transformations are given by the Poisson brackets with the Hermitian matrix $L = \ell \ell^\dagger$.

$$\delta_L^v \ell = -\frac{\lambda^{-2}}{\kappa} \{\text{Tr } V L, \ell\}, \quad \delta_L^v u = -\frac{\lambda^{-2}}{\kappa} \{\text{Tr } V L, u\}, \quad (55)$$

The generators of the rotations $\text{Tr } V L$ can be expanded explicitly as:

$$\kappa^{-1} \text{Tr } V L = \kappa^{-1} (2\epsilon_z \lambda^2 + \epsilon_- \lambda z + \epsilon_+ \lambda \bar{z}) = (2\epsilon_z (T_0 + T_z) + \epsilon_- T_+ + \epsilon_+ T_-).$$

The variations of ℓ and u under a right infinitesimal $\text{SU}(2)$ transformations are given instead by the Poisson brackets with the Hermitian matrix $\tilde{L}^{op} = \tilde{\ell}^\dagger \tilde{\ell}$.

$$\delta_R^{(v)} u = \frac{\tilde{\lambda}^2}{\kappa} \{\text{Tr } V (\tilde{L}^{op})^{-1}, u\}, \quad \delta_R^{(v)} \ell = \frac{\tilde{\lambda}^2}{\kappa} \{\text{Tr } V (\tilde{L}^{op})^{-1}, \ell\} = 0. \quad (56)$$

III. LATTICE WITH HEISENBERG DOUBLE AND CONSTRAINTS

A. Graphs, ribbons and phase spaces

In the standard LQG formalism, we associate to an edge of a graph the elements of the $T^*\text{SU}(2)$ phase space, i.e. the holonomy h and the fluxes \mathbf{X} and $\tilde{\mathbf{X}}$ (related to one another by the holonomy). In the new formulation, we are going to associate to an edge an element D of \mathcal{D}_+ . To take into account the fact that this group element can be decomposed into two ways, either $D = \ell u$ or $D = \tilde{u} \tilde{\ell}$, we are going to fatten the edge into a ribbon³ with orientations on its boundary, as described in Fig. 1. This naturally encodes the fact that $\ell u = \tilde{u} \tilde{\ell}$ on each edge.



FIG. 1: The ribbon is orientated such that $\ell u = \tilde{u} \tilde{\ell}$. The solid lines are the strands and the dashed lines represent the thickened source and target points of each edge. Flipping the box is not allowed, only rotations in the plane are authorized.

Note that since we deal with an orientable manifold, it is well known that a cell decomposition of a 2D surface can be represented as a ribbon graph. The 2D cell decomposition is therefore defined as a ribbon graph, with ribbon edges incident on ribbon vertices. Unlike the standard graph, we now have holonomies for both $\text{SU}(2)$ and $\text{SB}(2, \mathbb{C})$. The $\text{SU}(2)$ elements u, \tilde{u} are on the side of the ribbon edges, while the $\text{SB}(2, \mathbb{C})$ elements $\ell, \tilde{\ell}$ lie where the ribbon edges are glued to the ribbon vertices. The boundaries of the ribbon vertices therefore carry "translations holonomies", i.e. ℓ and/or ℓ^{-1} , as illustrated in Figure 2.

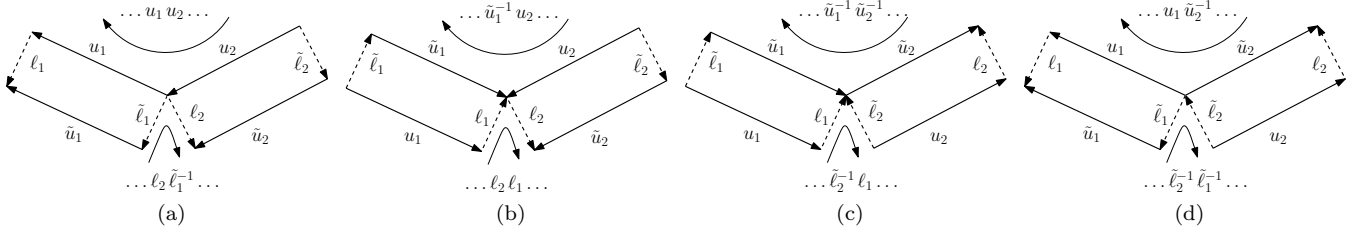


FIG. 2: The four possible ways to glue two ribbon edges together. The product of the $\text{SU}(2)$ elements along the solid lines always only contains some u and/or \tilde{u}^{-1} . The product of elements along the dashed lines always only contains some ℓ and/or $\tilde{\ell}^{-1}$.

With this formulation, we can easily generalize the construction from $T^*\text{SU}(2) \sim \text{ISO}(3)$ to $\text{SL}(2, \mathbb{C})$. This is not affecting the ribbon structure, the only thing that really changes is that ℓ is now an element of the non-Abelian group $\text{SB}(2, \mathbb{C})$.

B. The constraints

Now that we have detailed the Poisson structure on a single edge, we want to build a dynamics on a graph, made of first class constraints which generate the (infinitesimal) rotation and translation transformations and which can be used to describe discrete hyperbolic geometries on the graph.

In the graph, there are two types of "faces": the faces of the cell decomposition, whose boundary edges are u and \tilde{u} , and the ribbon vertices, whose boundary edges carry some ℓ and $\tilde{\ell}$, both represented in the Figure 3. Notice that they are four ways to glue two ribbon edges at a vertex as displayed in the Figure 2.

³ We would like to thank L. Freidel for sharing his idea regarding the possible use of the ribbon formalism in LQG.

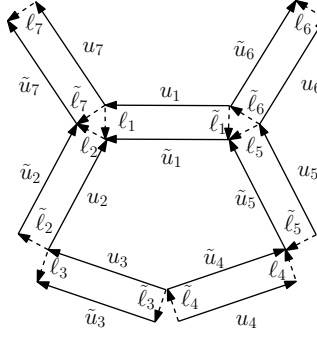


FIG. 3: Part of a ribbon graph. There is a closed face in bold lines, with holonomy $\tilde{u}_4^{-1}\tilde{u}_5^{-1}\tilde{u}_1^{-1}u_2u_3$. There are two closed ribbon vertices with holonomies $\ell_1\ell_2\tilde{\ell}_7^{-1}$ and $\ell_5\tilde{\ell}_1^{-1}\tilde{\ell}_6^{-1}$ in dashed lines.

As a consequence, the product of the $SU(2)$ elements around a face only contains some u and/or \tilde{u}^{-1} (or some u^{-1} and/or \tilde{u} upon reversing the face orientation, but never some u and u^{-1} for instance). Similarly, the product of $SB(2, \mathbb{C})$ elements around a ribbon vertex only contains some ℓ and/or $\tilde{\ell}^{-1}$ (or some ℓ^{-1} and/or $\tilde{\ell}$ depending on the orientation, but never some ℓ and ℓ^{-1} for example).

In addition to the constraint $\ell u = \tilde{u} \tilde{\ell}$ on each edge, we propose the following set of constraints,

$$\mathcal{G}_v \equiv \mathcal{L}_1 \cdots \mathcal{L}_{N_v} = \mathbf{1}, \quad \text{with } \mathcal{L}_i = \ell_i \text{ or } \tilde{\ell}_i^{-1}, \quad (57)$$

$$\mathcal{C}_f \equiv \mathcal{U}_1 \cdots \mathcal{U}_{N_f} = \mathbf{1}, \quad \text{with } \mathcal{U}_i = u_i \text{ or } \tilde{u}_i^{-1}, \quad (58)$$

where each $\mathcal{L}_i = \ell_i$ or $\tilde{\ell}_i^{-1}$ is the $SB(2, \mathbb{C})$ or \mathbb{R}^3 element on the leg $i = 1, \dots, N_v$ around the ribbon vertex v (while the order matters, the choice of the first edge does not), and \mathcal{U}_i is the $SU(2)$ element on the edge $i = 1, \dots, N_f$ around the face f . For instance, on the top of the Figure 3, there are two closed ribbon vertices, one with $\tilde{\ell}_7^{-1}\ell_2\ell_1$ and the other with $\tilde{\ell}_1^{-1}\ell_5\tilde{\ell}_6^{-1}$, and there is one face with $\tilde{u}_4^{-1}\tilde{u}_5^{-1}\tilde{u}_1^{-1}u_2u_3$.

We will call \mathcal{G} the *Gauss law* and \mathcal{C} the *flatness constraint*. We now show that \mathcal{G} generates $SU(2)$ transformations (hence the name) and that \mathcal{C} generates the (deformed) translations.

C. The Gauss law \mathcal{G}_v as generator of $SU(2)$ transformations

Consider for simplicity a N -valent ribbon vertex whose incident edges are oriented inward. The Gauss law then reads

$$\mathcal{G} = \ell_1 \cdots \ell_N. \quad (59)$$

In the $ISO(3)$ case, where $\ell = (\mathbf{1}, \mathbf{X})$, this product of matrices leads to the usual constraint of vanishing total angular momentum,

$$\mathcal{G} = \mathbf{1} \Leftrightarrow \sum \mathbf{X}_i = \mathbf{0}. \quad (60)$$

Hence we recover the usual the Gauss constraint and we know that it generates the local (at each vertex) $SU(2)$ transformations.

Let us consider the new case, when $\ell_i \in SB(2, \mathbb{C})$. The product $\prod_i \ell_i$ of the triangular matrices ℓ_i is easy to perform,

$$\mathcal{G} = \ell_1 \cdots \ell_N = \begin{pmatrix} \sum_{i=1}^N [\prod_{j=1}^{i-1} \lambda_j^{-1}] \prod_{i=1}^N \lambda_i & 0 \\ \sum_{i=1}^N [\prod_{j=1}^{i-1} \lambda_j^{-1}] z_i [\prod_{k=i+1}^N \lambda_k] & \prod_{i=1}^N \lambda_i^{-1} \end{pmatrix}, \quad (61)$$

and therefore the constraints are

$$\left. \begin{aligned} \mathcal{G} &= \ell_1 \cdots \ell_N = \mathbf{1} \\ \mathcal{G}^{-1\dagger} &= l_1 \cdots l_N = \mathbf{1} \end{aligned} \right\} \Leftrightarrow \begin{cases} \sum_{i=1}^N [\prod_{j=1}^{i-1} \lambda_j^{-1}] \prod_{i=1}^N \lambda_i = 1 \\ \sum_{i=1}^N [\prod_{j=1}^{i-1} \lambda_j^{-1}] z_i [\prod_{k=i+1}^N \lambda_k] = 0 \\ \sum_{i=1}^N [\prod_{j=1}^{i-1} \lambda_j^{-1}] \bar{z}_i [\prod_{k=i+1}^N \lambda_k] = 0. \end{cases} \quad (62)$$

The brackets between the matrix elements of \mathcal{G} and an arbitrary function f are

$$\{\mathcal{G}_{11}, f\} = \left\{ \prod_{i=1}^N \lambda_i, f \right\} = \prod_{i=1}^N \lambda_i \sum_{k=1}^N \lambda_k^{-1} \{\lambda_k, f\} \quad (63)$$

$$\{\mathcal{G}_{22}, f\} = \left\{ \prod_{i=1}^N \lambda_i^{-1}, f \right\} = - \prod_{i=1}^N \lambda_i^{-1} \sum_{k=1}^N \lambda_k^{-1} \{\lambda_k, f\} = - \prod_{i=1}^N \lambda_i^2 \{\mathcal{G}_{11}, f\} \quad (64)$$

$$\{\mathcal{G}_{21}, f\} = \prod_{j=1}^N \lambda_j^{-1} \sum_{i=1}^N \left\{ \lambda_i z_i \prod_{k=i+1}^N \lambda_k^2, f \right\} + \sum_{i=1}^N \lambda_i z_i \prod_{k=i+1}^N \lambda_k^2 \left\{ \prod_{j=1}^N \lambda_j^{-1}, f \right\}, \quad (65)$$

and similarly with \mathcal{G}^\dagger . It is convenient to express those brackets in terms of the transformation generated by \mathcal{GG}^\dagger ,

$$\prod_{k=1}^N \lambda_k^{-2} \{\text{Tr } V \mathcal{GG}^\dagger, f\} \equiv \epsilon_z \delta_z f + \epsilon_- \delta_- f + \epsilon_+ \delta_+ f, \quad (66)$$

where $V = \begin{pmatrix} 2\epsilon_z & \epsilon_- \\ \epsilon_+ & 0 \end{pmatrix}$. A direct calculation leads to

$$\text{Tr } V \mathcal{GG}^\dagger = 2\epsilon_z \prod_{i=1}^N \lambda_i^2 + \epsilon_- \sum_{i=1}^N \lambda_i z_i \prod_{j=i+1}^N \lambda_j^2 + \epsilon_+ \sum_{i=1}^N \lambda_i \bar{z}_i \prod_{j=i+1}^N \lambda_j^2, \quad (67)$$

therefore

$$\delta_z f = \prod_{k=1}^N \lambda_k^{-2} \left\{ 2 \prod_{i=1}^N \lambda_i^2, f \right\} = 2 \sum_{i=1}^N \lambda_i^{-2} \{\lambda_i^2, f\} = 4 \sum_{i=1}^N \lambda_i^{-1} \{\lambda_i, f\}, \quad (68)$$

$$\delta_- f = \prod_{k=1}^N \lambda_k^{-2} \left\{ \sum_{i=1}^N \lambda_i z_i \prod_{j=i+1}^N \lambda_j^2, f \right\}, \quad \delta_+ f = \prod_{k=1}^N \lambda_k^{-2} \left\{ \sum_{i=1}^N \lambda_i \bar{z}_i \prod_{j=i+1}^N \lambda_j^2, f \right\}. \quad (69)$$

We get

$$\{\mathcal{G}_{11}, f\} = \frac{1}{4} \left[\prod_{k=1}^N \lambda_k \right] \delta_z f = \frac{1}{4} \mathcal{G}_{11} \delta_z f \quad (70)$$

$$\{\mathcal{G}_{21}, f\} = \left[\prod_{k=1}^N \lambda_k \right] \delta_- f - \frac{1}{4} \left[\sum_{i=1}^N \prod_{k=1}^{i-1} \lambda_k^{-1} z_i \prod_{j=i+1}^N \lambda_j \right] \delta_z f = \mathcal{G}_{11} \delta_- f - \frac{1}{4} \mathcal{G}_{21} \delta_z f. \quad (71)$$

Notice that in the simple case $N = 1$, the bracket (66) generates the $\text{SU}(2)$ transformations on the phase space $\text{SL}(2, \mathbb{C})$. In order to prove that $\mathcal{G}, \mathcal{G}^\dagger$ generates $\text{SU}(2)$ rotations on $N > 1$ legs, we have to match the brackets with some linear combination of brackets with the generators $\ell_k \ell_k^\dagger$ of $\text{SU}(2)$ transformations on the leg k .

Proposition III.1. *The Gauss constraint generates $\text{SU}(2)$ transformations with braided parameter $V^{(k-1)}$ on the leg $k = 1, \dots, N$, i.e.*

$$\prod_{k=1}^N \lambda_k^{-2} \{\text{Tr } V \mathcal{GG}^\dagger, f\} = \sum_{k=1}^N \lambda_k^{-2} \text{Tr } V^{(k-1)} \{\ell_k \ell_k^\dagger, f\}, \quad (72)$$

where $V^{(k)}$ is defined by induction, $V^{(k)} = \lambda_k^{-2} \ell_k^\dagger V^{(k-1)} \ell_k$, and $V^{(0)} = V = \begin{pmatrix} 2\epsilon_z & \epsilon_- \\ \epsilon_+ & 0 \end{pmatrix}$.

Proof. The braided parameter is found to be

$$V^{(k)} = \left[\prod_{i=1}^k \lambda_i^{-2} \right] \ell_k^\dagger \cdots \ell_1^\dagger V \ell_1 \cdots \ell_k = \begin{pmatrix} 2\epsilon_z^{(k)} & \epsilon_-^{(k)} \\ \epsilon_+^{(k)} & 0 \end{pmatrix}, \quad \text{with} \quad \begin{cases} \epsilon_z^{(k)} = \epsilon_z + \frac{1}{2} \sum_{i=1}^k \left[\prod_{j=1}^i \lambda_j^{-2} \right] (\epsilon_- \lambda_i z_i + \epsilon_+ \lambda_i \bar{z}_i) \\ \epsilon_\pm^{(k)} = \left[\prod_{i=1}^k \lambda_i^{-2} \right] \epsilon_\pm \end{cases} \quad (73)$$

It can be noted that $V^{(N)} = V$ on-shell, i.e. when $\mathcal{G} = \mathcal{G}^\dagger = \mathbf{1}$. First consider the $SU(2)$ transformations along z ,

$$\sum_{k=1}^N \lambda_k^{-2} \text{Tr} V^{(k-1)} \{ \ell_k \ell_k^\dagger, f \}_{|\epsilon_\pm=0} = 2\epsilon_z \sum_{k=1}^N \lambda_k^{-2} \{ \lambda_k^2, f \} = \epsilon_z \delta_z f, \quad (74)$$

by direct comparison with (68). Next we set $\epsilon_z = \epsilon_+ = 0$ and focus on the variations with ϵ_- ,

$$\sum_{k=1}^N \lambda_k^{-2} \text{Tr} V^{(k-1)} \{ \ell_k \ell_k^\dagger, f \}_{|\epsilon_z=\epsilon_+=0} = \epsilon_- \sum_{k=2}^N \lambda_k^{-2} \sum_{i=1}^{k-1} \left[\prod_{j=1}^i \lambda_j^{-2} \right] \lambda_i z_i \{ \lambda_k^2, f \} + \epsilon_- \sum_{k=1}^N \lambda_k^{-2} \left[\prod_{j=1}^{k-1} \lambda_j^{-2} \right] \{ \lambda_k z_k, f \}. \quad (75)$$

The first term of the right hand side is

$$\begin{aligned} \epsilon_- \sum_{k=2}^N \lambda_k^{-2} \sum_{i=1}^{k-1} \left[\prod_{j=1}^i \lambda_j^{-2} \right] \lambda_i z_i \{ \lambda_k^2, f \} &= \epsilon_- \prod_{j=1}^N \lambda_j^{-2} \sum_{i=1}^{N-1} \lambda_i z_i \left[\prod_{p=i+1}^N \lambda_p^2 \right] \sum_{k=i+1}^N \lambda_k^{-2} \{ \lambda_k^2, f \} \\ &= \epsilon_- \prod_{j=1}^N \lambda_j^{-2} \sum_{i=1}^N \lambda_i z_i \left\{ \prod_{k=i+1}^N \lambda_k^2, f \right\}, \end{aligned} \quad (76)$$

where the empty product is 1. The second term writes

$$\epsilon_- \sum_{k=1}^N \lambda_k^{-2} \left[\prod_{j=1}^{k-1} \lambda_j^{-2} \right] \{ \lambda_k z_k, f \} = \epsilon_- \prod_{j=1}^N \lambda_j^{-2} \sum_{k=1}^N \left[\prod_{p=k+1}^N \lambda_p^2 \right] \{ \lambda_k z_k, f \}. \quad (77)$$

Therefore we get

$$\sum_{k=1}^N \lambda_k^{-2} \text{Tr} V^{(k-1)} \{ \ell_k \ell_k^\dagger, f \}_{|\epsilon_z=\epsilon_+=0} = \epsilon_- \left[\prod_{j=1}^N \lambda_j^{-2} \right] \sum_{i=1}^N \left\{ \lambda_i z_i \prod_{p=i+1}^N \lambda_p^2, f \right\} = \epsilon_- \delta_- f, \quad (78)$$

in agreement with (69). The variation with ϵ_+ works similarly.

D. The flatness constraints as generator of the (deformed) translations

We consider for simplicity a face whose N boundary edges have the same orientation as the face so that the flatness constraint reads

$$\mathcal{C} = u_N \cdots u_1. \quad (79)$$

Again, in the $ISO(3)$ case, we know that this constraint generates the Abelian translational symmetry. So we focus instead on the $SL(2, \mathbb{C})$ case.

Consider the transformation generated by the following bracket with \mathcal{C} on an arbitrary function f ,

$$\delta_M f \equiv \text{Tr} \delta M \mathcal{C}^{-1} \{ \mathcal{C}, f \}, \quad \text{with } \delta M = \begin{pmatrix} 2\epsilon_3 & \epsilon_- \\ \epsilon_+ & -2\epsilon_3 \end{pmatrix}. \quad (80)$$

It is equivalent to a translation on all the edges around the face (the translation on a single edge is generated by $\text{Tr} \delta M u^{-1} \{ u, \cdot \}$, as stated in the Proposition II.3).

Proposition III.2. *The flatness constraint generates $SB(2, \mathbb{C})$ transformations with braided parameter $\delta M^{(k-1)}$ on the edge $k = 1, \dots, N$,*

$$\text{Tr} \delta M \mathcal{C}^{-1} \{ \mathcal{C}, f \} = \sum_{k=1}^N \text{Tr} \delta M^{(k-1)} u_k^{-1} \{ u_k, f \}. \quad (81)$$

where $\delta M^{(k-1)} = u_{k-1} \cdots u_1 \delta M u_1^{-1} \cdots u_{k-1}^{-1}$.

Proof. The calculation is straightforward,

$$\begin{aligned} \text{Tr} \delta M \mathcal{C}^{-1} \{ \mathcal{C}, f \} &= \sum_{k=1}^N \text{Tr} u_{k-1} \cdots u_1 \delta M \mathcal{C}^{-1} u_N \cdots u_{k+1} \{ u_k, f \} \\ &= \sum_{k=1}^N \text{Tr} \left[u_{k-1} \cdots u_1 \delta M u_1^{-1} \cdots u_{k-1}^{-1} \right] u_k^{-1} \{ u_k, f \} = \sum_{k=1}^N \text{Tr} \delta M^{(k-1)} u_k^{-1} \{ u_k, f \}. \end{aligned} \quad (82)$$

E. A first class constraint algebra

Interestingly, the proof that the constraints form a first class algebra will not depend on the choice of phase space, ISO(3) or SL(2, C). Using Section II, the left rotation given by an element $v \in \text{SU}(2)$ acting at a ribbon vertex reads

$$\begin{aligned} \ell_1 u_1 &\mapsto v \ell_1 u_1 = (v \ell_1 (v^{(\ell_1)})^{-1}) ((v^{(\ell_1)}) u_1) = {}^{(v_1)} \ell_1 {}^{(v_1)} u_1, \text{ with } v_1 \equiv v^{(\ell_1)} \\ \ell_2 u_2 &\mapsto (v^{(\ell_1)}) \ell_2 u_2 = ((v^{(\ell_1)}) \ell_2 (v^{(\ell_1 \ell_2)})^{-1}) ((v^{(\ell_1 \ell_2)}) u_2) = {}^{(v_2)} \ell_2 {}^{(v_2)} u_2, \text{ with } v_2 \equiv v^{(\ell_1 \ell_2)} \\ \ell_N u_N &\mapsto v_{N-1} \ell_N u_N = (v_{N-1} \ell_N v_N^{-1}) (v_N u_N) = {}^{(v_N)} \ell_N {}^{(v_N)} u_N, \text{ with } v_N \equiv v^{(\ell_1 \ell_2 \dots \ell_N)}. \end{aligned} \quad (83)$$

Note that we have used the fact that there is a (possibly trivial) right action \triangleright of $G^* = \text{SB}(2, \mathbb{C}), \mathbb{R}^3$ on $G = \text{SU}(2)$, so that we have

$$v \triangleleft \ell_1 = v \ell_1 \equiv v^{(\ell_1)}, \quad v^{(\ell_1)} \ell_2 = (v \triangleleft \ell_1) \triangleleft \ell_2 = v \triangleleft (\ell_1 \ell_2) \equiv v^{(\ell_1 \ell_2)}, \quad v \in G, \ell_i \in G^*. \quad (84)$$

Let us compute the transformation of the Gauss law under v (again in the case the incident edges are oriented inward),

$${}^{(v_1)} \ell_1 \dots {}^{(v_N)} \ell_N = (v \ell_1 v_1^{-1}) (v_1 \ell_2 v_2^{-1}) \dots (v_{N-1} \ell_N v_N^{-1}) = v \ell_1 \dots \ell_N v_N^{-1}. \quad (85)$$

Thus when the Gauss law is satisfied, $\ell_1 \dots \ell_N = \mathbf{1}$, we know that $v_N = v$ which implies that the transformed triangular matrices still satisfy the Gauss law, ${}^{(v_1)} \ell_1 \dots {}^{(v_N)} \ell_N = \mathbf{1}$.

As the Gauss law generates SU(2) transformations, we expect the brackets of the Gauss law with itself to vanish on-shell. One can directly check that those constraints form a first class system,

$$\{\mathcal{G}_1, \mathcal{G}_2\} = -[r, \mathcal{G}_1 \mathcal{G}_2], \quad \{\mathcal{G}_1^{-1\dagger}, \mathcal{G}_2^{-1\dagger}\} = -[r^\dagger, \mathcal{G}_1^{-1\dagger} \mathcal{G}_2^{-1\dagger}], \quad \{\mathcal{G}_1, \mathcal{G}_2^{-1\dagger}\} = -[r^\dagger, \mathcal{G}_1 \mathcal{G}_2^{-1\dagger}]. \quad (86)$$

Let us check the brackets in the case one edge is outgoing, and to simplify we consider a bivalent vertex (this generalizes straightforwardly, though quite inconveniently in terms of notations). We have $\mathcal{G} = \ell_\alpha \tilde{\ell}_\beta^{-1}$ where α, β are the two edges. The bracket reads

$$\begin{aligned} \{\mathcal{G}_1, \mathcal{G}_2\} &= \{\ell_{\alpha 1} \tilde{\ell}_{\beta 1}^{-1}, \ell_{\alpha 2} \tilde{\ell}_{\beta 2}^{-1}\} = -\ell_{\alpha 1} \ell_{\alpha 2} [r^\dagger, \tilde{\ell}_{\beta 1}^{-1} \tilde{\ell}_{\beta 2}^{-1}] - [r, \ell_{\alpha 1} \ell_{\alpha 2}] \tilde{\ell}_{\beta 1}^{-1} \tilde{\ell}_{\beta 2}^{-1} \\ &= -[r, \ell_{\alpha 1} \tilde{\ell}_{\beta 1}^{-1} \ell_{\alpha 2} \tilde{\ell}_{\beta 2}^{-1}] = -[r, \mathcal{G}_1 \mathcal{G}_2]. \end{aligned} \quad (87)$$

We have used that $r - r^\dagger$ is the Casimir to go from the first to the second line.

Furthermore, the brackets between the Gauss constraints associated to two ribbon vertices v, v' connected by an edge e vanish. Indeed, if \mathcal{G}_v contains ℓ_e , then $\mathcal{G}_{v'}$ contains $\tilde{\ell}_e$ (or vice versa), and moreover $\{\ell, \tilde{\ell}\} = 0$. This way, we can conclude that the algebra generated by the Gauss law is first class, whatever the edge orientations are.

Now we consider the bracket between the Gauss law at a vertex v and the flatness constraint around a face f such that v is a vertex of f . There are several situations depending on the orientations of the two edges of f that meet at v . Let us look at the case where the edge α is incoming at v while the edge β is outgoing. The only non-trivial part of the bracket has the following form

$$\{\ell_{\alpha 1} \tilde{\ell}_{\beta 1}^{-1}, u_{\beta 2} u_{\alpha 2}\} = \ell_{\alpha 1} \{\tilde{\ell}_{\beta 1}^{-1}, u_{\beta 2}\} u_{\alpha 2} + u_{\beta 2} \{\ell_{\alpha 1}, u_{\alpha 2}\} \tilde{\ell}_{\beta 1}^{-1} = 0. \quad (88)$$

The fact that the bracket vanishes identically reflects the fact that the flatness constraint is SU(2) invariant. Indeed, if D_α transform on the left (at v) with $v \in \text{SU}(2)$, then

$$u_\alpha \mapsto v^{(\ell_1)} u_\alpha, \quad (89)$$

and D_β transforms on the right (since the edge is oriented outward) with the modified parameter $(v^{(\ell_1)})^{-1}$ that has transited through D_α . More generally we get

$$\{\mathcal{G}_1, \mathcal{C}_2\} = 0. \quad (90)$$

The bracket of the flatness constraint with itself on the same face gives (in the case it only contains some u , $\mathcal{C} = u_N \dots u_1$)

$$\{\mathcal{C}_1, \mathcal{C}_2\} = [r^\dagger, \mathcal{C}_1 \mathcal{C}_2]. \quad (91)$$

The last case to check involves the two faces f, f' shared by an edge. However, if f contains the variable u associated to that edge then f' contains \tilde{u} and not u (or vice versa), and due to $\{u, \tilde{u}\} = 0$, the bracket between the two flatness constraint vanishes.

IV. GEOMETRIC MEANING OF THE CONSTRAINTS

A. Gauss constraint and cosine laws

1. ISO(3) case and the "flat" cosine law

In the flat case, the Gauss law we have introduced in (57), realized on a 3-valent vertex with incident edges, boils down to

$$\ell_1 \ell_2 \ell_3 = \mathbf{1} \rightsquigarrow \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 = \vec{0}. \quad (92)$$

The 3-valent vertex is combinatorially dual to a triangle and \mathbf{X}_e is interpreted as the normal to the edge dual to e , in the plane spanned by the triangle. This provides the well known geometric interpretation that the triangle dual to the vertex geometrically closes: this is the *closure constraint*.

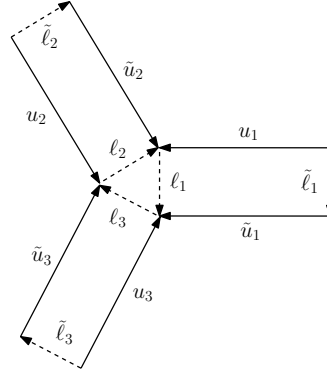


FIG. 4: A 3-valent ribbon vertex with its incident edges inward. The Gauss law is $\mathcal{G} = \ell_1 \ell_2 \ell_3 = \mathbf{1}$.

Equivalently, the Gauss law corresponds to the three SU(2) invariant equations,

$$\text{Tr} \ell_j^\dagger \ell_i^\dagger \ell_i \ell_j = \text{Tr}(\ell_k \ell_k^\dagger)^{-1} \Leftrightarrow (\mathbf{X}_i + \mathbf{X}_j)^2 = \mathbf{X}_k^2, \text{ for } i, j, k = 1, 2, 3 \text{ all different.} \quad (93)$$

By expanding the square, it comes

$$|\mathbf{X}_i|^2 + |\mathbf{X}_j|^2 - 2 |\mathbf{X}_i| |\mathbf{X}_j| \cos \phi_{ij} = |\mathbf{X}_k|^2, \text{ with } \cos \phi_{ij} \equiv -\mathbf{X}_i \cdot \mathbf{X}_j / (|\mathbf{X}_i| |\mathbf{X}_j|). \quad (94)$$

These relations describe the angles ϕ_{ij} of the triangles in terms of the edge lengths $|\mathbf{X}_e|$. We have just rederived the well known fact that the Gauss constraint encodes in the flat case the closure constraint that is equivalent to the three *cosine laws*.

Trading ISO(3) for SL(2, \mathbb{C}) the expressions (93) will generalize to the curved case. The difficulty is however to identify the analogue of the normal vectors \mathbf{X}_e .

2. SL(2, \mathbb{C}) case and the hyperbolic cosine law

Beside the Iwasawa decomposition, the Cartan decomposition of SL(2, \mathbb{C}) is also available. It will provide the key to identify the analogue of the normal vectors. This decomposition states that any element can be written as the product of a boost and a rotation. When dealing with the momentum variable ℓ , we have

$$\ell = B h^{-1}, h \in \text{SU}(2), \quad B = e^{-\vec{\sigma} \cdot \vec{b}},$$

where $B \in \text{SL}(2, \mathbb{C})$ is a pure boost in the Lorentz group, that is uniquely characterized as a 2×2 Hermitian matrix satisfying $B = B^\dagger$ and $\det B = 1$. The main advantage is that the set of pure boosts is stable under conjugation by SU(2) group elements, unlike triangular matrices. Moreover, B and ℓ define the same vector $\mathbf{T} = \text{Tr} \ell \ell^\dagger \vec{\sigma} / (2\kappa)$, since

$$L = D D^\dagger = \ell \ell^\dagger = B B^\dagger.$$

It will also be useful to have the explicit formula for the boost $B = \cosh b \mathbf{1} - \sinh b (\hat{\mathbf{b}} \cdot \vec{\sigma})$ in its spinorial form, with rapidity $2b$ and boost direction $\hat{\mathbf{b}}$ expressed in terms of λ and z :

$$B = \begin{pmatrix} \cosh(b) - \sinh(b) \hat{b}_z & -\sinh(b) \hat{b}_- \\ -\sinh(b) \hat{b}_+ & \cosh(b) + \sinh(b) \hat{b}_z \end{pmatrix} \quad (95)$$

with $\cosh(2b) = \frac{1 + \lambda^2(\lambda^2 + |z|^2)}{2\lambda^2}$, $\hat{b}_+ \sinh(2b) = -\lambda z$, $\sinh(2b) \hat{b}_z = \frac{\lambda^{-2} + |z|^2 - \lambda^2}{2}$.

Let us also remark that hBh^{-1} , which is still a pure boost and which enters the definition of $\mathbf{T}^{op} = \frac{1}{2\kappa} \text{Tr}(\ell^\dagger \ell \vec{\sigma}) = \frac{1}{2\kappa} \text{Tr}(hB^\dagger Bh^{-1} \vec{\sigma})$, has the simple expression

$$hBh^{-1} = \frac{1}{\sqrt{2 + 2\kappa T_0}} (\mathbf{1} + \ell^\dagger \ell) = \frac{1}{\sqrt{2 + 2\kappa T_0}} (\mathbf{1} + L^{op}) \quad (96)$$

We want to show now that the Gauss law on a 3-valent vertex contains exactly the information to construct a *hyperbolic* triangle. Note that unlike the flat case, a hyperbolic triangle t is totally specified by its three angles or its three lengths.

We consider three incoming edges, so that $\mathcal{G} = \ell_1 \ell_2 \ell_3$, as in the Figure 4. We use the Cartan decomposition $\ell = Bh^{-1}$, with B a boost and h a rotation, to get

$$\ell_1 \ell_2 \ell_3 = \mathfrak{B}_1 \mathfrak{B}_2 \mathfrak{B}_3 H^{-1} = \mathbf{1}, \quad (97)$$

with

$$\mathfrak{B}_1 = B_1, \quad \mathfrak{B}_2 = h_1^{-1} B_2 h_1, \quad \mathfrak{B}_3 = h_1^{-1} h_2^{-1} B_3 h_2 h_1, \quad H = h_3 h_2 h_1. \quad (98)$$

We take one \mathfrak{B}_i to the right hand side, say $\mathfrak{B}_1 \mathfrak{B}_2 = H \mathfrak{B}_3^{-1}$, and multiply on the left by the adjoint equation to get rid of the rotation H , which gives

$$\ell_2^\dagger \ell_1^\dagger \ell_1 \ell_2 = (\ell_3 \ell_3^\dagger)^{-1} \Leftrightarrow \mathfrak{B}_2 \mathfrak{B}_1^2 \mathfrak{B}_2 = (\mathfrak{B}_3^{-1})^2 \quad (99)$$

By taking the trace, we get the following three equations,

$$\text{Tr} \mathfrak{B}_1^2 \mathfrak{B}_2^2 = \text{Tr} \mathfrak{B}_3^2, \quad \text{Tr} \mathfrak{B}_2^2 \mathfrak{B}_3^2 = \text{Tr} \mathfrak{B}_1^2, \quad \text{Tr} \mathfrak{B}_3^2 H^{-1} \mathfrak{B}_1^2 H = \text{Tr} \mathfrak{B}_2^2. \quad (100)$$

By writing

$$\mathfrak{B} = \cosh b/2 \mathbf{1} + \sinh b/2 \hat{\mathbf{b}} \cdot \vec{\sigma}, \quad (101)$$

where $\hat{\mathbf{b}}$ is the normalized direction of the boost, we get explicitly

$$\begin{aligned} \cosh b_1 \cosh b_2 + \sinh b_1 \sinh b_2 \hat{\mathbf{b}}_1 \cdot R(h_1^{-1}) \hat{\mathbf{b}}_2 &= \cosh b_3, \\ \cosh b_2 \cosh b_3 + \sinh b_2 \sinh b_3 \hat{\mathbf{b}}_2 \cdot R(h_2^{-1}) \hat{\mathbf{b}}_3 &= \cosh b_1, \\ \cosh b_3 \cosh b_1 + \sinh b_3 \sinh b_1 \hat{\mathbf{b}}_3 \cdot R(h_3^{-1}) \hat{\mathbf{b}}_1 &= \cosh b_2. \end{aligned} \quad (102)$$

Here $R(h)$ denotes the rotation h in the vector representation, i.e. a 3D rotation. There is a nice way to rewrite the scalar products $\hat{\mathbf{b}}_i \cdot R(h_i^{-1}) \hat{\mathbf{b}}_j$ in terms of the vector variables $\mathbf{T}, \mathbf{T}^{op}$. First note that $\hat{\mathbf{b}}$ is the normalized vector \mathbf{T} , and $R(h) \hat{\mathbf{b}}$ is the normalized \mathbf{T}^{op} ,

$$\sinh b \hat{\mathbf{b}} = \frac{1}{2} \text{Tr} B^2 \vec{\sigma} = \frac{1}{2} \text{Tr} \ell \ell^\dagger \vec{\sigma} = \mathbf{T}, \quad (103)$$

$$\sinh b R(h) \hat{\mathbf{b}} = \frac{1}{2} \text{Tr} B^2 h^{-1} \vec{\sigma} h = \frac{1}{2} \text{Tr} \ell^\dagger \ell \vec{\sigma} = \mathbf{T}^{op}, \quad (104)$$

i.e. the rotation h maps \mathbf{T} to \mathbf{T}^{op} . This suggests to define $\hat{\mathbf{b}}^{op} = \mathbf{T}^{op}/|\mathbf{T}^{op}|$, such that $\hat{\mathbf{b}}^{op} = R(h) \hat{\mathbf{b}}$, and

$$\cos \phi_{ij} \equiv -\hat{\mathbf{b}}_i^{op} \cdot \hat{\mathbf{b}}_j, \quad (ij) = (12), (23), (31), \quad (105)$$

so that ϕ_{ij} is the angle between i and j in the hyperbolic triangle defined by the lengths $\kappa b_1, \kappa b_2, \kappa b_3$. Indeed, the equations (102) take the form of the hyperbolic cosine law

$$\cosh b_i \cosh b_j - \sinh b_i \sinh b_j \cos \phi_{ij} = \cosh b_k, \quad (106)$$

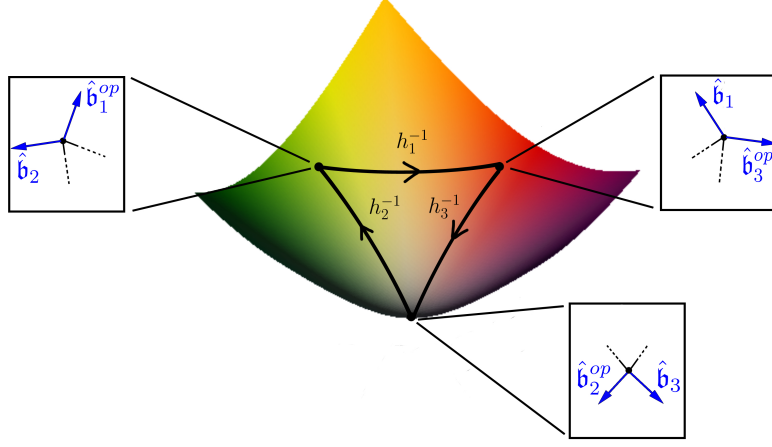


FIG. 5: This represents the way the vectors $\hat{b}_e, \hat{b}_e^{op}$ are assigned to the dashed lines of the ribbon vertex.

where i, j, k are such that $\epsilon_{ijk} = 1$. The ribbon vertex corresponds to a triangle whose boundary is composed of oriented dashed edges carrying ℓ_1, ℓ_2, ℓ_3 . The angles ϕ_{ij} are associated to the corners of this triangle. On the dashed line i , we associate \hat{b}_i to its target vertex and \hat{b}_i^{op} to its source vertex, and the rotation h_i^{-1} maps the former to the latter. This is summarized in the Figure 5. \hat{b}_i^{op} is interpreted as the normal vector to the edge i at the meeting point of i and j , and \hat{b} as the normal vector at the meeting point of i and k (still with the convention $\epsilon_{ijk} = 1$).

If one edge e incident to the ribbon vertex is outgoing, we get $\tilde{\ell}_e^{-1}$ in the Gauss law. By writing $\tilde{\ell}^{-1} = \tilde{B}\tilde{h}^{-1}$, we define

$$-\sinh \tilde{b} \hat{b} = \frac{1}{2} \text{Tr} \tilde{B}^{-2} \tilde{h}^{-1} \tilde{\sigma} \tilde{h} = \frac{1}{2} \text{Tr} \tilde{\ell} \tilde{\ell}^\dagger \tilde{\sigma} = \tilde{\mathbf{T}}, \quad (107)$$

$$-\sinh \tilde{b} R(\tilde{h}^{-1}) \hat{b}^{op} = \frac{1}{2} \text{Tr} \tilde{B}^{-2} \tilde{\sigma} = \frac{1}{2} \text{Tr} \tilde{\ell}^\dagger \tilde{\ell} \tilde{\sigma} = \tilde{\mathbf{T}}^{op}, \quad (108)$$

and then find that the angles ϕ_{ij} are evaluated through the scalar products with \hat{b}_e instead of \hat{b}_e at the target vertex of the dashed line $\tilde{\ell}_e$ and with \hat{b}_e^{op} instead of \hat{b}_e^{op} at the source vertex of the dashed line. Moreover, the relation between both is now

$$\hat{b}^{op} = R(\tilde{h}^{-1}) \hat{b}. \quad (109)$$

B. Dihedral angles between hyperbolic triangles and the flatness constraint

In the flat case, the geometric interpretation of the flatness constraint was described in details in [12]. We focus here on the new curved case, and show that similar results can be obtained. That is, we want to show that the flatness constraint enables to evaluate the extrinsic curvature (measured by dihedral angles between triangles) as the one of a homogeneous 3D hyperbolic geometry made by gluing hyperbolic triangles. For simplicity, we restrict our attention to the case of three triangles glued together. In the dual picture, this corresponds to three ribbon vertices, connected by three edges e_1, e_2, e_6 which represent the edges of the triangles meeting at a common point. This point is represented as the face formed by those three ribbon edges. The notations and orientations are fixed by the Figure 6.

The 2D angles around the face are

$$\cos \phi_{26} = -\hat{b}_2^{op} \cdot \hat{b}_6, \quad \cos \phi_{16} = -\hat{b}_1 \cdot \hat{b}_6, \quad \cos \phi_{12} = -\hat{b}_1 \cdot \hat{b}_2^{op}. \quad (110)$$

If the gluing of triangles can be embedded in 3D hyperbolic space, the dihedral angle between the two triangles which meet along the edge dual to e_1 is

$$\cos \theta_1 = \frac{\cos \phi_{26} - \cos \phi_{12} \cos \phi_{16}}{\sin \phi_{12} \sin \phi_{16}}. \quad (111)$$

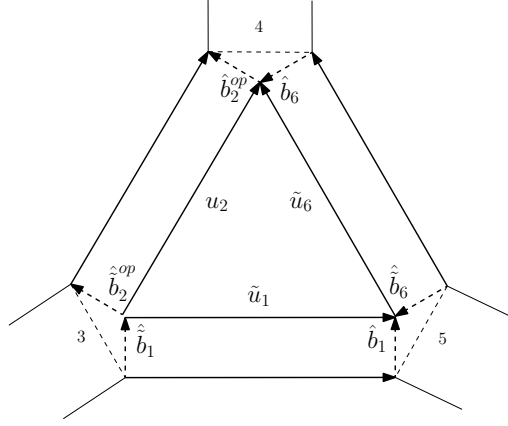


FIG. 6: This is a face with three edges, and the flatness constraint $\mathcal{C} = u_1 \tilde{u}_1^{-1} \tilde{u}_6^{-1}$. It is dual to a vertex shared by three triangles, which meet along the edges dual to e_1, e_2, e_6 . We have indicated the vectors $\hat{\mathbf{b}}_e, \hat{\mathbf{b}}_e^{op}, \hat{\mathbf{b}}_e, \hat{\mathbf{b}}_e^{op}$ which are relevant to the 2d angles.

This would fix the extrinsic curvature as a function of the intrinsic geometry, determined by the 2D angles or equivalently the lengths.

On the other hand, we have an independent notion of dihedral angles on our phase space. Introduce the normals to the two triangles hinged on the edge dual to e_1 , at the point where the three triangles meet,

$$N_{12} = \frac{\hat{\mathbf{b}}_1 \times \hat{\mathbf{b}}_2^{op}}{|\hat{\mathbf{b}}_1 \times \hat{\mathbf{b}}_2^{op}|} = \frac{\hat{\mathbf{b}}_1 \times \hat{\mathbf{b}}_2^{op}}{\sin \phi_{12}}, \quad N_{16} = \frac{\hat{\mathbf{b}}_6 \times \hat{\mathbf{b}}_1}{|\hat{\mathbf{b}}_6 \times \hat{\mathbf{b}}_1|} = \frac{\hat{\mathbf{b}}_6 \times \hat{\mathbf{b}}_1}{\sin \phi_{16}}. \quad (112)$$

The dihedral angle is the scalar product between these two normals, but since they are defined in the frame of two different triangles (one uses $\hat{\mathbf{b}}_1$ and the other uses $\hat{\mathbf{b}}_1$), it is necessary to transport N_{16} along e_1 using the rotation $R(\tilde{u}_1^{-1})$. This leads to the definition

$$\cos \Theta_1 = -N_{12} \cdot R(\tilde{u}_1^{-1}) N_{16}. \quad (113)$$

Using the fact that $R(\tilde{u}_1^{-1}) \hat{\mathbf{b}}_1 = -\hat{\mathbf{b}}_1$, a direct calculation shows that

$$\begin{aligned} \cos \Theta_1 &= -\frac{(\hat{\mathbf{b}}_1 \times \hat{\mathbf{b}}_2^{op}) \cdot (R(\tilde{u}_1^{-1}) \hat{\mathbf{b}}_6 \times R(\tilde{u}_1^{-1}) \hat{\mathbf{b}}_1)}{\sin \phi_{12} \sin \phi_{16}}, \\ &= -\frac{(\hat{\mathbf{b}}_1 \cdot R(\tilde{u}_1^{-1}) \hat{\mathbf{b}}_6) (\hat{\mathbf{b}}_2^{op} \cdot R(\tilde{u}_1^{-1}) \hat{\mathbf{b}}_1) - (\hat{\mathbf{b}}_2^{op} \cdot R(\tilde{u}_1^{-1}) \hat{\mathbf{b}}_6) (\hat{\mathbf{b}}_1 \cdot R(\tilde{u}_1^{-1}) \hat{\mathbf{b}}_1)}{\sin \phi_{12} \sin \phi_{16}}, \\ &= -\frac{\cos \phi_{16} \cos \phi_{12} + \hat{\mathbf{b}}_2^{op} \cdot R(\tilde{u}_1^{-1}) \hat{\mathbf{b}}_6}{\sin \phi_{12} \sin \phi_{16}} \end{aligned} \quad (114)$$

Now we are in position to relate the two above notions of dihedral angles to the flatness constraint. The latter reads $\mathcal{C} = u_2 \tilde{u}_1^{-1} \tilde{u}_6^{-1}$. We consider the matrix element of \mathcal{C} in the vector representation contracted with $\hat{\mathbf{b}}_2^{op}$ and $\hat{\mathbf{b}}_6$,

$$H_1 = -\hat{\mathbf{b}}_2^{op} \cdot R(\mathcal{C}) \hat{\mathbf{b}}_6 + \hat{\mathbf{b}}_2^{op} \cdot \hat{\mathbf{b}}_6. \quad (115)$$

$H_1 = 0$ on-shell and it is one of the three independent components of $\mathcal{C} - \mathbf{1}$. Using $R(\tilde{u}_6^{-1}) \hat{\mathbf{b}}_6 = -\hat{\mathbf{b}}_6$ and $R(u_2^{-1}) \hat{\mathbf{b}}_2^{op} = -\hat{\mathbf{b}}_2^{op}$, we get

$$\begin{aligned} H_1 &= -\hat{\mathbf{b}}_2^{op} \cdot R(\tilde{u}_1^{-1}) \hat{\mathbf{b}}_6 - \cos \phi_{26}, \\ &= \sin \phi_{12} \sin \phi_{16} \cos \Theta_1 + \cos \phi_{12} \cos \phi_{16} - \cos \phi_{26}, \\ &= \sin \phi_{12} \sin \phi_{16} (\cos \Theta_1 - \cos \theta_1). \end{aligned} \quad (116)$$

To get to the second line, we have used (114), and to get the last line we have recognized $\cos \theta_1$ as defined in (111).

Therefore we see that the constraint $\mathcal{C} = \mathbf{1}$ forces the holonomy around the face, $u_2 \tilde{u}_1^{-1} \tilde{u}_6^{-1}$, to know about the hyperbolic dihedral angles, and identifies the extrinsic curvature as measured by $\cos \Theta_1$ with the geometry of 3D hyperbolic space determined by the intrinsic geometry of the triangles (measured by $\cos \theta_1$ as a function of the 2D angles).

V. SOLUTIONS OF THE CONSTRAINTS

We show in this section that the theory is topological, in the sense that the reduced phase space, i.e. the set of solutions of the constraints, only depends on the topology and not on the choice of cell decomposition. We proceed in a way completely analogous to the way the moduli space of flat connections is identified as the set of solutions in BF theory (2D Yang-Mills at weak coupling). In fact since both cases $\text{ISO}(3)$ and $\text{SL}(2, \mathbb{C})$ are expressed in the same formalism, the proof for the flat case extends in a direct manner to the curved case, once we have re-expressed the variables in terms of the Heisenberg double variables. Let us consider the $\text{SL}(2, \mathbb{C})$ case since it is new.

Just as in the flat case, first we have to gauge fix the $(\text{SU}(2))$ rotational and $(\text{SB}(2, \mathbb{C}))$ deformed translational symmetries (up to global residual symmetries), then solve the constraints on as many faces and vertices as possible, until we arrive at equations which only depend on the topology.

We consider the graph dual to a triangulation of a surface of genus g , with V, E, F denoting the sets of vertices, edges and faces. The set E of edges can be partitioned into 3 sub-sets: a spanning tree \mathcal{T} in the graph (with $|V| - 1$ edges), a set of edges \mathcal{T}^* dual to a spanning tree in the triangulation (with $|F| - 1 = |E| - |V| + 1 - 2g$ edges), and a set of $2g$ “crossing” edges CE .

The gauge fixing of the $\text{SU}(2)$ symmetry consists of setting $u_e = \mathbb{I}$ for $e \in \mathcal{T}$. This implies

$$\forall e \in \mathcal{T}, \quad u_e = \tilde{u}_e = \mathbb{I}, \quad \ell_e = \tilde{\ell}_e. \quad (117)$$

The gauge fixing of the $\text{SB}(2, \mathbb{C})$ symmetry is similar in the dual: we take $\ell_e = \mathbb{I}$ for $e \in \mathcal{T}^*$ so that

$$\forall e \in \mathcal{T}^*, \quad \ell_e = \tilde{\ell}_e = \mathbb{I}, \quad u_e = \tilde{u}_e. \quad (118)$$

These two gauge fixings are consistent with one another because the condition (117) is left invariant under a $\text{SB}(2, \mathbb{C})$ transformation (the transformed $u^{(m)}$ is defined by $u m = {}^{(u)}m u^{(m)}$ which means ${}^{(u)}m = m$ and $u^{(m)} = \mathbf{1}$ when $m = \mathbf{1}$), and the condition (118) is left invariant by $\text{SU}(2)$ transformations. They can be seen graphically as a retract of the ribbon lines carrying the $\text{SU}(2)$ elements for the edges in \mathcal{T} (the solid lines in the Figure 1 are reduced to two points) and a thinning of the ribbon edges in \mathcal{T}^* so that they become regular edges (and the dashed lines in the Figure 1 disappear).

We choose arbitrary roots for \mathcal{T} and the spanning tree dual to \mathcal{T}^* in the triangulation. This induces a natural orientation of their edges from the leaves towards the root. There is then a canonical procedure to solve the constraints.

Let us start with the flatness constraints. The vertices of the spanning tree in the triangulation are dual to faces of the ribbon graph and we denote $F_{\mathcal{T}^*}$ this set of faces. We can solve the set of constraints $\{\mathcal{C}_f\}_{f \in F_{\mathcal{T}^*}}$ except for the face dual to the root. Indeed, each edge in \mathcal{T}^* is shared by exactly two faces, one being the source and the other the target with respect to the orientation induced by the root. Moreover, each such edge carries a single $\text{SU}(2)$ element since $u_e = \tilde{u}_e$, which appears in the flatness constraints of the source and target faces only. Moreover, u_e only appears once in those constraints. We can therefore use the constraint at the source face to express $u_e = f(\{u_\alpha\})$ as a product of other $\text{SU}(2)$ elements. This expression is then injected in the constraint at the target face. We say that this way u_e has been eliminated. Starting from the faces dual to the leaves, down to the root face, we see that all $\text{SU}(2)$ elements carried by the edges in \mathcal{T}^* are eliminated, since they are expressed as products of the only $\text{SU}(2)$ elements we are left with, i.e. those on the edges in CE . Graphically, this elimination process simply consists in removing the edges of \mathcal{T}^* which merges all faces together until only one remains, with the ribbon edges of CE on the boundary. It is called a *deletion* process.

We proceed similarly for the Gauss constraints. Each edge $e \in \mathcal{T}$ carries a single $\text{SB}(2, \mathbb{C})$ element since $\ell_e = \tilde{\ell}_e$, which appears in exactly two Gauss constraints, those at the source and target vertices e is incident to. We solve the Gauss constraints along \mathcal{T} , from its leaves down to its root vertex. At each step we use the constraint at the source vertex to express ℓ_e as a product of other elements which is then injected in the constraint at the target vertex. All $\{\ell_e\}_{e \in \mathcal{T}}$ are thus expressed as functions on $\{\ell_e\}_{e \in CE}$. One is left with the constraint at the root vertex on the remaining variables. Graphically, this process contracts the ribbon edges of \mathcal{T} so as to merge the ribbon vertices they are incident to. This is called a *contraction* process.

The sequence of contractions and deletions we described above is well known and turns the ribbon graph we started with in a “rosette” which has a single ribbon vertex, $2g$ ribbon edges and a single face, as shown in the Figure 7.

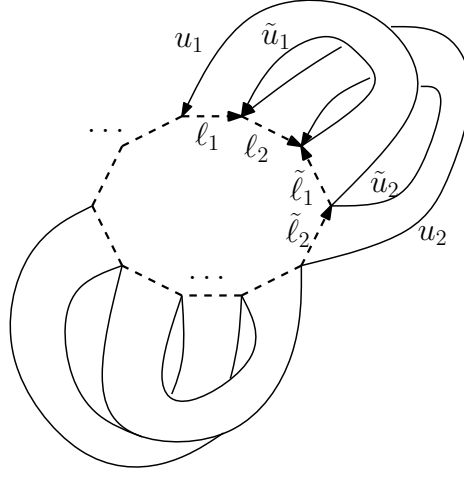


FIG. 7: The rosette graph, with a single ribbon vertex, $2g$ ribbon edges and a single face.

From this rosette it is trivial to read the remaining constraints,

$$\mathcal{C} = u_1 \tilde{u}_2^{-1} \tilde{u}_1^{-1} u_2 \cdots u_{2g-1} \tilde{u}_{2g}^{-1} \tilde{u}_{2g-1}^{-1} u_{2g} = \mathbb{I}, \quad (119)$$

$$\mathcal{G} = \tilde{\ell}_{2g}^{-1} \tilde{\ell}_{2g-1}^{-1} \ell_{2g} \ell_{2g-1} \cdots \tilde{\ell}_2^{-1} \tilde{\ell}_1^{-1} \ell_2 \ell_1 = \mathbb{I}. \quad (120)$$

We end up with them no matter which triangulation we started with and they clearly depend on the genus only, meaning that we have successfully defined a topological theory.

Conclusion & Outlook

We have constructed topological models describing flat and hyperbolic discrete geometries. In the flat case, we have recalled the standard discretization of BF theory and we have reformulated it in the less standard formalism of Heisenberg double. This reformulation has allowed us to construct in a direct manner a new phase space, namely $SL(2, \mathbb{C})$, as a deformation of the Heisenberg double of the flat case. The new $SL(2, \mathbb{C})$ phase space is constructed from the classical r -matrix of $\mathfrak{sl}(2, \mathbb{C})$ and is interpreted as a deformation of the flat $ISO(3)$ case, with deformed notions of rotations and translations. This phase space can be viewed as a $SU(2)$ configuration space and a curved momentum space $SB(2, \mathbb{C})$ (identified as the hyperboloid of time-like normalized vectors in Minkowski space-time).

Considering a ribbon decomposition of the 2D surface, provided with one copy of $SL(2, \mathbb{C})$ on each edge, we introduced closure constraints on vertices and flatness constraints on faces. On the one hand, these constraints generate rotations and $SB(2, \mathbb{C})$ translations; on the other hand, they have a natural geometrical meaning: on the trivalent graph dual to a 2D triangulation, the closure constraint at each vertex (dual to a triangle) imposes the hyperbolic cosine law on the triangle, and the flatness constraint on each face (dual to a point) ensures that these triangles are consistently glued in the sense that the triangulation can be embedded in a homogeneous hyperbolic geometry. From this perspective, we have showed how this new model describes discrete hyperbolic geometries. We have further proved that this model is topological, in a way similar to discretized BF theory.

This new phase space and model open up many different routes to explore.

Link with continuum limit: In the flat case, we know that we have considered a discretization of the BF theory with $\Lambda = 0$. We would like to show that similarly the new model corresponds to a discretization of BF theory with $\Lambda < 0$ (of the form $BF + \Lambda B^3$). This would ensure that we are indeed describing 3D Riemannian gravity with negative cosmological constant. It would require understanding how to derive the $SL(2, \mathbb{C})$ phase space from the continuous triad and connection fields of BF theory. This is currently under investigation.

Quantization: Once again in the flat case, we know that the quantization of this model gives rise to LQG which can be precisely related to the Ponzano-Regge model. In the $SL(2, \mathbb{C})$ case, we would expect to obtain LQG and the Turaev-Viro model defined in terms of $\mathcal{U}_q(\mathfrak{su}(2))$ (with q real though). Some elements of the quantization process already exist. For example in [20], the quantum analogue of the components of ℓ are given, with $q = e^{\hbar\kappa}$

$$\ell \rightarrow \hat{\ell} = \begin{pmatrix} K^{-1} & 0 \\ -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_+ & K \end{pmatrix} \quad l \rightarrow \hat{l} = \begin{pmatrix} K & (q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_- \\ 0 & K^{-1} \end{pmatrix}. \quad (121)$$

Hence it seems natural to identify $\lambda \rightarrow K^{-1} = q^{-J_z/2}$, $z \rightarrow -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_+$, $\bar{z} \rightarrow -(q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_-$, where we recognize the usual $\mathcal{U}_q(\mathfrak{su}(2))$ generators. The quantum version R of the classical r -matrix becomes

$$r = \frac{i\kappa}{4} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & 4 & -1 & \\ & & & 1 \end{pmatrix} \rightsquigarrow R = q^{-\frac{1}{4}} \begin{pmatrix} q^{\frac{1}{2}} & & & \\ & 1 & & \\ & q^{\frac{1}{2}} - q^{-\frac{1}{2}} & 1 & \\ & & & q^{\frac{1}{2}} \end{pmatrix} \sim \mathbf{1} - i\hbar r. \quad (122)$$

The quantum version of the Poisson bracket is then given by

$$R^\dagger \hat{\ell}_1 \hat{\ell}_2 = \hat{\ell}_2 \hat{\ell}_1 R^\dagger, \quad R^\dagger \hat{l}_1 \hat{l}_2 = \hat{l}_2 \hat{l}_1 R^\dagger, \quad R^\dagger \hat{l}_1 \hat{\ell}_2 = \hat{\ell}_2 \hat{l}_1 R^\dagger, \quad (123)$$

which give then the $\mathcal{U}_q(\mathfrak{su}(2))$ commutation relations

$$K J_\pm K^{-1} = q^{\pm \frac{1}{2}} J_\pm, \quad [J_+, J_-] = \frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}}. \quad (124)$$

The precise quantization procedure is currently under investigation.

Braiding: In the quantum group case such as for $\mathcal{U}_q(\mathfrak{su}(2))$, the tensor product becomes "non-commutative" which leads to the notion of braiding. This braiding is characterized by the quantum R -matrix. In the construction we have proposed here, there are different braidings involved. We have seen that both the translations and the rotations have a different type of braiding in Proposition II.2 and II.3. We have also seen that the vector $\hat{\mathbf{b}}$ was constructed from the boosts, using a different braiding (cf (98)). It would be interesting to see how these braidings relate together and to the $\mathcal{U}_q(\mathfrak{su}(2))$ braiding used to construct the observables in the LQG context [10, 11]. A priori the braiding of the symmetries is likely to be related to the braiding of the Drinfeld double $U_q(\mathrm{SL}(2, \mathbb{C}))$. The quantum analogue of the braiding for the $\hat{\mathbf{b}}$ is less clear.

Link with Chern-Simons: It is well known that a BF theory with a cosmological constant can be rewritten as a Chern-Simons theory [6, 22, 23]. The discretization of the phase space approach to Chern-Simons theory is well-understood and is given by the Fock-Rosly formalism [21]. In the flat case $\Lambda = 0$, the link between the discretized BF theory and the Fock-Rosly formalism was given in [4]. It would be interesting to see how their approach can be generalized to our construction.

Curved twisted geometries: Spin networks defined with gauge group $\mathrm{SU}(2)$ can be interpreted as the quantization of a special type of discrete geometries, the *twisted geometries* [24]. The generalization of this framework to the curved case would be extremely interesting. Another useful tool, related to the twisted geometries, is their parametrization in terms of spinor variables [25–27]. The quantum group version of this approach has been described in details in [11]. It would be interesting to explore what are the classical analogue of these spinor operators and how they could be used to parametrize the $\mathrm{SL}(2, \mathbb{C})$ phase space. Essentially one would like to define the variables that transforms as spinors or vectors under the action of the non trivial Poisson Lie group $\mathrm{SU}(2)$. This is currently under investigation.

Other signatures and signs for Λ : For simplicity we have focused here on the Euclidian case with $\Lambda \leq 0$. The other combinations are also of interest. For example, the Euclidian case with $\Lambda > 0$ would be interesting to explore to get discrete spherical geometries. It would lead supposedly to the quantum group $\mathcal{U}_q(\mathfrak{su}(2))$ with q root of unity. In this case, it is well known that this quantum group is not a Hopf algebra but a quasi-Hopf algebra. Hence, at the classical level, we would expect to deal with quasi-bialgebras [29]. This is currently under investigation.

Insights for the 4D case? The fact that the cosmological constant is implemented using a quantum group in a 3D space-time has been used as a motivation to do the same thing for a 4D space-time [30–33]. The proof of this conjecture is a long standing problem. Thanks to our model we have now some new tools to explore if this proposal is actually justified. Indeed, we can expect that unlocking the link between our model with the continuum model in 3D will shed some new light on how the kinematical space is affected by $\Lambda \neq 0$.

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Appendix A: The ISO(3) case

Generators and r -matrix: The $\mathfrak{so}(3)$ generators (in the spin 1 representation) can be written as 3×3 anti-symmetric matrices:

$$j_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad j_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad j_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A1})$$

They satisfy the following commutation relations: $[j_i, j_j] = \epsilon_{ij}^k j_k$. The generators of the Abelian Lie algebra \mathbb{R}^3 are 3-vectors:

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (\text{A2})$$

We can write the generators of the classical double of $\mathfrak{so}(3)$ using 4×4 matrices:

$$J_i = \begin{pmatrix} j_i & \mathbf{0} \\ 0 & 0 \end{pmatrix}, \quad E^i = \begin{pmatrix} 0_{3 \times 3} & \mathbf{e}^i \\ 0 & 0 \end{pmatrix}. \quad (\text{A3})$$

Then the r -matrix is given by

$$r = \sum_i E^i \otimes J_i = \begin{pmatrix} 0 & 0 & 0 & J_1 \\ 0 & 0 & 0 & J_2 \\ 0 & 0 & 0 & J_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad r^t = \sum_i J_i \otimes E^i = \begin{pmatrix} 0 & E^3 & -E^2 & 0 \\ -E^3 & 0 & E^1 & 0 \\ E^2 & -E^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A4})$$

They are 16×16 matrices written by blocks of 4×4 matrices. If $\mathfrak{so}(3)$ with generators J_i is equipped with the trivial cocycle $\delta = 0$ then the Abelian Lie algebra \mathbb{R}^3 is its dual Lie algebra. Its associate cocycle is $\delta(E^k) = \epsilon_{ij}^k E^i \otimes E^j$. To completely precise the structures of the classical double $\mathfrak{d}_0(\mathfrak{so}(3))$, we need to introduce the \mathfrak{d}_0 -invariant bilinear form which characterizes the fact that E^i is the dual of J_i : $\langle E^i, J_j \rangle = \delta_j^i$, $\langle J_i, J_j \rangle = \langle E^i, E^j \rangle = 0$.

The Heisenberg double and Poisson bracket structure: Let $D \in \text{ISO}(3)$, its left decomposition is given by

$$D = (\mathbf{R}, \mathbf{X}) = \ell u, \quad \ell = \begin{pmatrix} \mathbb{I}_{3 \times 3} & \mathbf{X} \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ 0 & 1 \end{pmatrix}, \quad (\text{A5})$$

where \mathbf{X} is a 3-vector and \mathbf{R} is a 3×3 rotation matrix. Introducing the notation $D_1 = D \otimes \mathbb{I}$ and $D_2 = \mathbb{I} \otimes D$, its commutation relations, for ISO(3) seen as a Heisenberg double, can be written as

$$\{D_1, D_2\} = -r D_1 D_2 + D_1 D_2 r^t. \quad (\text{A6})$$

which imply for the rotation and translation variables, u and ℓ respectively,

$$\{\ell_1, \ell_2\} = -[r, \ell_1 \ell_2], \quad \{\ell_1, u_2\} = -\ell_1 r u_2, \quad \{u_1, \ell_2\} = \ell_2 r^\dagger u_1, \quad \{u_1, u_2\} = -[r^\dagger, u_1 u_2]. \quad (\text{A7})$$

This gives as back the usual Poisson brackets of the flux and holonomy variables of LQG,

$$\{X_i, X_j\} = \epsilon_{ij}^k X_k, \quad \{X_i, \mathbf{R}\} = -j_i \mathbf{R}, \quad \{\mathbf{R}, \mathbf{R}\} = 0. \quad (\text{A8})$$

For the right Iwasawa decomposition of ISO(3), $D = \tilde{u} \tilde{\ell}$, we get,

$$\{\tilde{\ell}_1, \tilde{\ell}_2\} = -[r, \tilde{\ell}_1 \tilde{\ell}_2], \quad \{\tilde{\ell}_1, \tilde{u}_2\} = \tilde{u}_2 r \tilde{\ell}_1, \quad \{\tilde{u}_1, \tilde{\ell}_2\} = -\tilde{u}_1 r^t \tilde{\ell}_2, \quad \{\tilde{u}_1, \tilde{u}_2\} = -[r^t, \tilde{u}_1 \tilde{u}_2], \quad (\text{A9})$$

or equivalently,

$$\{\tilde{X}_i, \tilde{X}_j\} = \epsilon_{ij}^k \tilde{X}_k, \quad \{\tilde{X}_i, \tilde{\mathbf{R}}\} = \tilde{\mathbf{R}} j_i, \quad \{\tilde{\mathbf{R}}, \tilde{\mathbf{R}}\} = 0. \quad (\text{A10})$$

Appendix B: The $\text{SL}(2, \mathbb{C})$ case

Generators and r -matrix: The $\mathfrak{su}(2)$ generators (in the spin 1/2 representation) can be represented by the Pauli matrices $\vec{\sigma}$ satisfying $[\sigma_i, \sigma_j] = 2i\epsilon_{ij}^k \sigma_k$. Explicitly,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B1})$$

The generators of $\mathfrak{sb}(2, \mathbb{C})$ which generate the triangular matrix ℓ are modified boosts defined by

$$\kappa^{-1} \tau^i = i\sigma_i - \frac{i}{2} [\sigma_z, \sigma_i] = i\sigma_i + \epsilon_{zi}^k \sigma_k \quad (\text{B2})$$

$$\kappa^{-1} \tau^x = i\sigma_x + \sigma_y = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix}, \quad \kappa^{-1} \tau^y = i\sigma_y - \sigma_x = i\tau^x = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \quad \kappa^{-1} \tau^z = i\sigma^z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

They satisfy the commutation relations, $[\tau^i, \tau^j] = 2i\kappa(\delta_{jz}\tau^i - \delta_{iz}\tau^j)$.

The r -matrix then explicitly reads as,

$$r = \frac{1}{4} \sum_i \tau^i \otimes \sigma_i = \frac{i\kappa}{4} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & 4 & -1 & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad r^t = \frac{1}{4} \sum_i \sigma_i \otimes \tau^i = \frac{i\kappa}{4} \begin{pmatrix} 1 & & & \\ & -1 & 4 & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \quad (\text{B3})$$

If $\mathfrak{su}(2)$ with generators σ_i is equipped with the cocycle $\delta(\sigma_k) = 2i\kappa(\delta_k^i \delta_z^j - \delta_k^j \delta_z^i) \sigma_i \otimes \sigma_j$, then the Lie algebra $\mathfrak{sb}(2, \mathbb{C})$ is its dual algebra. Its associated cocycle is $\delta_*(\tau^k) = 2i\epsilon_{ij}^k \tau^i \otimes \tau^j$. To precise completely the structures of the classical double $\mathfrak{d}(\mathfrak{su}(2))$, we need to introduce the \mathfrak{d} -invariant bilinear form which characterizes the fact that τ^i is the dual of σ_i : $\langle \tau^i, \sigma_j \rangle = \delta_j^i$, $\langle \sigma_i, \sigma_j \rangle = \langle \tau^i, \tau^j \rangle = 0$.

The Heisenberg double and Poisson bracket structure: The phase space $\text{SL}(2, \mathbb{C})$ can be seen as a Heisenberg double. Then its symplectic structure is given by (18) for its left Iwasawa decomposition where $D = \ell u \in \text{SL}(2, \mathbb{C})$. We expand the $\text{SL}(2, \mathbb{C})$ Poisson brackets (19), (21) explicitly in terms of the components of the momentum variables ℓ , $l = (\ell^\dagger)^{-1}$ and configuration variables u .

- The brackets between components of the triangular matrix ℓ and ℓ^\dagger :

$$\{\lambda, z\} = \frac{i\kappa}{2} \lambda z, \quad \{\lambda, \bar{z}\} = -\frac{i\kappa}{2} \lambda \bar{z}, \quad \{\bar{z}, z\} = -i\kappa(\lambda^2 - \lambda^{-2}), \quad (\text{B4})$$

- The brackets between components of the unitary matrix u :

$$\begin{aligned} \{\alpha, \beta\} &= -\frac{i\kappa}{2} \alpha \beta, \quad \{\alpha, \bar{\beta}\} = -\frac{i\kappa}{2} \alpha \bar{\beta}, \quad \{\alpha, \bar{\alpha}\} = i\kappa |\beta|^2, \\ \{\bar{\alpha}, \beta\} &= \frac{i\kappa}{2} \bar{\alpha} \beta, \quad \{\bar{\alpha}, \bar{\beta}\} = \frac{i\kappa}{2} \bar{\alpha} \bar{\beta}, \quad \{\beta, \bar{\beta}\} = 0, \end{aligned} \quad (\text{B5})$$

- The brackets between ℓ or l and u :

$$\begin{aligned} \{\lambda, \alpha\} &= -\frac{i\kappa}{4} \lambda \alpha, & \{\lambda, \beta\} &= \frac{i\kappa}{4} \lambda \beta, & \{\lambda, \bar{\alpha}\} &= \frac{i\kappa}{4} \lambda \bar{\alpha}, & \{\lambda, \bar{\beta}\} &= -\frac{i\kappa}{4} \lambda \bar{\beta}, \\ \{\lambda^{-1}, \alpha\} &= \frac{i\kappa}{4} \lambda^{-1} \alpha, & \{\lambda^{-1}, \beta\} &= -\frac{i\kappa}{4} \lambda^{-1} \beta, & \{\lambda^{-1}, \bar{\alpha}\} &= -\frac{i\kappa}{4} \lambda^{-1} \bar{\alpha}, & \{\lambda^{-1}, \bar{\beta}\} &= \frac{i\kappa}{4} \lambda^{-1} \bar{\beta}, \\ \{z, \alpha\} &= -\frac{i\kappa}{4} (z\alpha + 4\lambda^{-1}\beta), & \{z, \beta\} &= \frac{i\kappa}{4} z\beta, & \{z, \bar{\alpha}\} &= \frac{i\kappa}{4} z\bar{\alpha}, & \{z, \bar{\beta}\} &= -\frac{i\kappa}{4} (z\bar{\beta} - 4\lambda^{-1}\bar{\alpha}), \\ \{\bar{z}, \alpha\} &= \frac{i\kappa}{4} (\bar{z}\alpha + 4\lambda^{-1}\bar{\beta}), & \{\bar{z}, \beta\} &= -\frac{i\kappa}{4} \bar{z}\beta, & \{\bar{z}, \bar{\alpha}\} &= -\frac{i\kappa}{4} \bar{z}\bar{\alpha}, & \{\bar{z}, \bar{\beta}\} &= -\frac{i\kappa}{4} (-\bar{z}\bar{\beta} + 4\lambda^{-1}\bar{\alpha}), \end{aligned} \quad (\text{B6})$$

Let us now use the relation between the right and left Iwasawa decompositions to determine the Poisson brackets between the ℓ , u variables of the left Iwasawa decomposition and the $\bar{\ell}$, \bar{u} variables of the right Iwasawa decomposition. Starting from $\ell u = \bar{u} \bar{\ell}$, we get rid of \bar{u} by considering $\bar{\ell}^\dagger \bar{\ell} = u^{-1} \ell^\dagger \ell u$, which gives

$$\begin{pmatrix} \tilde{\lambda}^2 + |\tilde{z}|^2 & \tilde{\lambda}^{-1} \tilde{z} \\ \tilde{\lambda}^{-1} \tilde{z} & \tilde{\lambda}^{-2} \end{pmatrix} = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \lambda^2 + |z|^2 & \lambda^{-1} z \\ \lambda^{-1} z & \lambda^{-2} \end{pmatrix} \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}. \quad (\text{B7})$$

We deduce that

$$\begin{aligned}\tilde{\lambda}^{-2} &= \lambda^{-2} + |\beta|^2(\lambda^2 - \lambda^{-2} + |z|^2) - \lambda^{-1}(\bar{\alpha}\beta\bar{z} + \alpha\bar{\beta}z) \\ \tilde{z} &= \tilde{\lambda} \left(-\alpha\beta(\lambda^2 - \lambda^{-2} + |z|^2) + \lambda^{-1}(\alpha^2 z - \beta^2 \bar{z}) \right).\end{aligned}\quad (\text{B8})$$

This allows to evaluate the Poisson brackets between $\tilde{\ell}$ and ℓ and u ,

$$\left\{ \tilde{\ell}_1^{-1}, u_2 \right\} = u_2 r \tilde{\ell}_1^{-1}, \quad \left\{ \tilde{\ell}_1, \ell_2 \right\} = 0. \quad (\text{B9})$$

From the first equality of the above equation, we can in particular infer that

$$\left\{ \tilde{\ell}_1, u_2 \right\} = -\tilde{\ell}_1 u_2 r^\dagger. \quad (\text{B10})$$

Finally, using $\tilde{u} = \ell u \tilde{\ell}^{-1}$, we get

$$\left\{ \tilde{u}_1, u_2 \right\} = 0, \quad \left\{ \tilde{u}_1, \ell_2 \right\} = -r^\dagger \tilde{u}_1 \ell_2. \quad (\text{B11})$$

In terms of the components, we have

- The brackets between $\tilde{\ell}$ and u :

$$\left\{ \tilde{\lambda}, \alpha \right\} = -i\frac{\kappa}{4} \alpha \tilde{\lambda}, \quad \left\{ \tilde{\lambda}, \bar{\alpha} \right\} = i\frac{\kappa}{4} \bar{\alpha} \tilde{\lambda}, \quad \left\{ \tilde{\lambda}, \beta \right\} = -i\frac{\kappa}{4} \beta \tilde{\lambda}, \quad \left\{ \tilde{\lambda}, \bar{\beta} \right\} = -i\frac{\kappa}{4} \bar{\beta} \tilde{\lambda}, \quad (\text{B12})$$

$$\left\{ \tilde{z}, \alpha \right\} = -i\frac{\kappa}{4} \alpha \tilde{z}, \quad \left\{ \tilde{z}, \bar{\alpha} \right\} = i\frac{\kappa}{4} \bar{\alpha} \tilde{z} - i\kappa \tilde{\lambda}^{-1} \beta, \quad \left\{ \tilde{z}, \beta \right\} = -i\frac{\kappa}{4} \beta \tilde{z}, \quad \left\{ \tilde{z}, \bar{\beta} \right\} = i\frac{\kappa}{4} \bar{\beta} \tilde{z} + i\kappa \tilde{\lambda}^{-1} \alpha. \quad (\text{B13})$$

- The brackets between $\tilde{\ell}^\dagger$ and u :

$$\left\{ \bar{\tilde{z}}, \alpha \right\} = -\frac{i\kappa}{4} (-4\bar{\beta}\tilde{\lambda}^{-1} + \bar{\tilde{z}}\alpha), \quad \left\{ \bar{\tilde{z}}, \bar{\beta} \right\} = \frac{i\kappa}{4} \bar{\tilde{z}}\beta, \quad \left\{ \bar{\tilde{z}}, \beta \right\} = -\frac{i\kappa}{4} (4\bar{\alpha}\tilde{\lambda}^{-1} + \beta\bar{\tilde{z}}), \quad \left\{ \bar{\tilde{z}}, \bar{\alpha} \right\} = \frac{i\kappa}{4} \bar{\tilde{z}}\bar{\alpha}. \quad (\text{B14})$$

Appendix C: Infinitesimal symmetries

Translations

We have the translations realized as

$$D \rightarrow m D = m \ell u = {}^{(m)}\ell {}^{(m)}u \rightarrow \left\{ \begin{array}{l} {}^{(m)}u = u \\ {}^{(m)}\ell = m \ell \end{array} \right., \quad D \rightarrow D m = \ell u m = \ell^{(m)} u^{(m)} \rightarrow \left\{ \begin{array}{l} {}^{(u)}m u^{(m)} = m u \\ \ell^{(m)} = \ell ({}^{(u)}m) \end{array} \right. \quad (\text{C1})$$

ISO(3) case, translation: In the case of the right translation for ISO(3), we have ${}^{(u)}m = u m u^{-1}$ and $u^{(m)} = u$. Setting $m = \mathbf{1} + \delta m = \mathbf{1} + \vec{\varepsilon} \cdot \vec{E}$, and plugging it into (C1), we obtain

$$\text{Infinitesimal left translations: } \left\{ \begin{array}{l} \delta_L^{(m)} u = 0 \\ \delta_L^{(m)} \ell = \delta m \ell \end{array} \right., \quad \text{Infinitesimal right translations: } \left\{ \begin{array}{l} \delta_R^{(m)} u = 0 \\ \delta_R^{(m)} \ell = \ell (u \delta m u^{-1}) \end{array} \right. \quad (\text{C2})$$

Let us calculate now the different Poisson brackets we proposed in Proposition II.2.

$$\delta_R^{(m)} u = \langle \delta m_1; u_1^{-1} \{u_1, u_2\} \rangle_1 = \langle \delta m_1; 0 \rangle_1 = 0, \quad (\text{C3})$$

$$\delta_R^{(m)} \ell = \langle \delta m_1; u_1^{-1} \{u_1, \ell_2\} \rangle_1 = \langle \delta m_1; u_1^{-1} \ell_2 r^t u_1 \rangle_1 = \sum_i \langle \vec{\varepsilon} \cdot \vec{E} \otimes \mathbf{1}; u^{-1} J_i u \otimes \ell E_i \rangle_1 = \ell (u \delta m u^{-1}) \quad (\text{C4})$$

$$\delta_L^{(m)} u = \langle \delta m_1; \{\tilde{u}_1, u_2\} \tilde{u}_1^{-1} \rangle_1 = \langle \delta m_1; 0 \rangle_1 = 0 \quad (\text{C5})$$

$$\delta_L^{(m)} \ell = \langle \delta m_1; \{\tilde{u}_1, \ell_2\} \tilde{u}_1^{-1} \rangle_1 = \langle \delta m_1; \ell_2 r^t \tilde{u}_1, \tilde{u}_1^{-1} \rangle_1 = \sum_i \langle \vec{\varepsilon} \cdot \vec{E} \otimes \mathbf{1}; J_i \otimes \ell E_i \rangle_1 = \delta m \ell \quad (\text{C6})$$

SL(2, \mathbb{C}) case, deformed translation: We write again $m = \mathbf{1} + \delta m \in \text{SB}(2, \mathbb{C})$, and furthermore introduce $M = m m^\dagger$, as well as $\delta M = \delta m + \delta m^\dagger$. We can actually express δm in terms of δM .

$$\delta m = \begin{pmatrix} \varepsilon_z & 0 \\ \varepsilon_+ & -\varepsilon_z \end{pmatrix}, \quad \delta M = \begin{pmatrix} 2\varepsilon_z & \varepsilon_- \\ \varepsilon_+ & -2\varepsilon_z \end{pmatrix}, \quad \frac{1}{2}[\delta M, \sigma_z] = \begin{pmatrix} 0 & -\varepsilon_- \\ \varepsilon_+ & 0 \end{pmatrix}, \quad \delta m = \frac{1}{2} \left(\delta M + \frac{1}{2}[\delta M, \sigma_z] \right). \quad (\text{C7})$$

We focus first on the right infinitesimal deformed translations. We deduce the twisted infinitesimal translation ${}^{(u)}\delta m$, coming from ${}^{(u)}m$ in (C1), using ${}^{(u)}M = u M u^{-1}$,

$${}^{(u)}\delta m = \frac{1}{2} \left(u \delta M u^{-1} + \frac{1}{2} [u \delta M u^{-1}, \sigma_z] \right). \quad (C8)$$

This allows to recover each of the transformations.

$$\delta_R^{(m)} u = u^{(m)} - u \sim (\mathbf{1} - {}^{(u)}\delta m) u (\mathbf{1} + \delta m) - u = \frac{1}{4} u [\delta M, \sigma_z - u^{-1} \sigma_z u], \quad (C9)$$

$$\delta_R^{(m)} \ell = \ell ({}^{(u)}\delta m) = \ell \frac{1}{2} \left(u \delta M u^{-1} + \frac{1}{2} [u \delta M u^{-1}, \sigma_z] \right). \quad (C10)$$

We would like now to relate the expression in terms of the Poisson brackets given in (II.3) to these infinitesimal transformations. For this we use once again the explicit expression Poisson brackets.

$$\text{Tr}_1 \delta M_1 u_1^{-1} \{u_1, u_2\} = \text{Tr}_1 \delta M_1 u_1^{-1} (r^\dagger u_1 u_2 - u_1 u_2 r^\dagger) = \sum_i \left(\text{Tr}(u \delta M u^{-1} \tau_i^\dagger) (\sigma_i u) - \text{Tr}(\delta M \tau_i^\dagger) (u \sigma_i) \right). \quad (C11)$$

Then coming back to the expression of τ_i in terms of the Pauli matrices, $\tau_i^\dagger = -i\kappa(\sigma_i + \frac{1}{2}[\sigma_z, \sigma_i])$, and using the identity $\sum_i \text{Tr}(A \sigma_i) \sigma_i = 2A - \text{Tr} A \mathbf{1}$, holding for arbitrary 2×2 matrices A , we have:

$$\sum_i \text{Tr}(A \tau_i^\dagger) \sigma_i = i\kappa(-2A - [A, \sigma_z] + \text{Tr} A). \quad (C12)$$

Putting the pieces together, we finally get:

$$\begin{aligned} \kappa^{-1} \text{Tr}_1 \delta M_1 u_1^{-1} \{u_1, u_2\} &= -\frac{i}{2} \left((u \delta M u^{-1} + \frac{1}{2} [u \delta M u^{-1}, \sigma_z]) u - u (\delta M + \frac{1}{2} [\delta M, \sigma_z]) \right) = i \frac{u}{4} [\delta M, \sigma_z - u^{-1} \sigma_z u], \\ \kappa^{-1} \text{Tr}_1 \delta M_1 u_1^{-1} \{u_1, \ell_2\} &= -\kappa^{-1} \text{Tr}_1 \delta M_1 u_1^{-1} \ell_2 r^\dagger u_1 = -\frac{1}{4\kappa} \ell \sum_i \text{Tr}(u \delta M u^{-1} \tau_i^\dagger) \sigma_i \\ &= i \frac{\ell}{2} \left(u \delta M u^{-1} + \frac{1}{2} [u \delta M u^{-1}, \sigma_z] \right), \end{aligned} \quad (C13)$$

which match respectively (C9) and (C10), modulo the factor $-i$. Let us consider now the infinitesimal left translations, which are much easier to determine, since we are using the left decomposition. By inspection of (C1), we have directly that

$$\delta_L^{(m)} u = 0, \quad \delta_L^{(m)} \ell = \delta m \ell. \quad (C14)$$

We now evaluate directly the Poisson brackets from Proposition II.3 (TO CHECK again)

$$\delta_L^{(m)} u = -\frac{i}{4\kappa} \text{Tr}_1 \delta M_1 \{\tilde{u}_1, u_2\} \tilde{u}_1^{-1} = -\frac{i}{4\kappa} \text{Tr}_1 \delta M_1 0 = 0, \quad (C15)$$

$$\delta_L^{(m)} \ell = -\frac{i}{4\kappa} \text{Tr}_1 \delta M_1 \{\tilde{u}_1, \ell_2\} \tilde{u}_1^{-1} = \frac{i}{4\kappa} \sum_i \text{Tr}(\delta M \tau_i^\dagger) \sigma_i \ell = \delta m \ell, \quad (C16)$$

where we used (C7) and (C12) in the last equation.

Rotations

The action of the rotations is given by

$$D \rightarrow D v = \ell u v = \ell^{(v)} u^{(v)} \rightarrow \begin{cases} u^{(v)} = u v \\ \ell^{(v)} = \ell \end{cases}, \quad D \rightarrow v D = v \ell u = {}^{(v)}\ell {}^{(v)}u \rightarrow \begin{cases} {}^{(v)}u = v^{(\ell)} u \\ {}^{(v)}\ell v^{(\ell)} = v \ell \end{cases} \quad (C17)$$

ISO(3) case, rotation: A quick inspection of (C17) with $v = \mathbf{1} + i\vec{\epsilon} \cdot \mathbf{J} = \mathbf{1} + V$ in mind leads to

$$\text{Infinitesimal left rotations: } \begin{cases} \delta_L^{(v)} u = V u \\ \delta_L^{(v)} \ell = V \ell - \ell V \end{cases}, \quad \text{Infinitesimal right rotations: } \begin{cases} \delta_R^{(v)} u = u V \\ \delta_R^{(v)} \ell = 0 \end{cases}. \quad (C18)$$

Let us calculate now the different Poisson brackets we proposed in Proposition II.4.

$$\delta_L^v \ell = -\langle V_1; \ell_1^{-1} \{ \ell_1, \ell_2 \} \rangle_1 = \langle V_1; \ell_1^{-1} (r \ell_1 \ell_2 - \ell_1 \ell_2 r) \rangle_1 = \sum_i \langle \vec{\varepsilon} \cdot \vec{J}; (\ell^{-1} E_i \ell \otimes J_i \ell - E_i \otimes \ell J_i) \rangle_1 = V \ell - \ell V, \quad (\text{C19})$$

$$\delta_L^v u = -\langle V_1; \ell_1^{-1} \{ \ell_1, u_2 \} \rangle_1 = \langle V_1; \ell_1^{-1} \ell_1 r u_2 \rangle_1 = \sum_i \langle \vec{\varepsilon} \cdot \vec{J}; E_i \otimes J_i u \rangle_1 = V u. \quad (\text{C20})$$

$$\delta_R^v \ell = \langle V_1; \tilde{\ell}_1^{-1} \{ \tilde{\ell}_1, \ell_2 \} \rangle_1 = \langle V_1; 0 \rangle_1 = 0, \quad (\text{C21})$$

$$\delta_R^v u = \langle V_1; \tilde{\ell}_1^{-1} \{ \tilde{\ell}_1, u_2 \} \rangle_1 = \langle V_1; \tilde{\ell}_1^{-1} u_2 r \tilde{\ell}_1 \rangle_1 = \sum_i \langle \vec{\varepsilon} \cdot \vec{J}; E_i \otimes u J_i \rangle_1 = u V, \quad (\text{C22})$$

SL(2, \mathbb{C}) case, rotation: We are going to plug in $v = \mathbf{1} + i\vec{\varepsilon} \cdot \vec{\sigma}$ into (C17) to get the action of the left infinitesimal rotation. To study the transformation of ℓ under such infinitesimal rotation, it is actually convenient to look at how L transforms since $L \rightarrow v L v^{-1}$. Recalling the expressions for ℓ and L ,

$$\ell = \begin{pmatrix} \lambda & 0 \\ z & \lambda^{-1} \end{pmatrix}, \quad L = \begin{pmatrix} \lambda^2 & \lambda \bar{z} \\ \lambda z & \lambda^{-2} + |z|^2 \end{pmatrix},$$

we explicitly parameterize the SU(2) matrix v and we get the parameters $^{(v)}\lambda$ and $^{(v)}z$ of $^{(v)}\ell$ as functions of ℓ and v :

$$v = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \Rightarrow \begin{cases} ^{(v)}\lambda^2 &= |\alpha|^2 \lambda^2 + |\beta|^2 (\lambda^{-2} + |z|^2) - \bar{\alpha} \bar{\beta} \lambda z - \alpha \beta \lambda \bar{z} \\ ^{(v)}\lambda ^{(v)}z &= \bar{\alpha}^2 \lambda z - \beta^2 \lambda \bar{z} + \bar{\alpha} \beta (\lambda^2 - \lambda^{-2} - |z|^2) \end{cases}$$

Replacing v by its infinitesimal version, we get at leading order in ε the components of $\delta_L^{(v)} \ell$:

$$\delta_L^{(v)} \lambda = \lambda^{(v)} - \lambda \sim \frac{i}{2} (\varepsilon_- z - \varepsilon_+ \bar{z}), \quad \delta_L^{(v)} z = z^{(v)} - z \sim -2i \varepsilon_z z - \frac{i}{2\lambda} z (\varepsilon_- z - \varepsilon_+ \bar{z}) + \frac{i}{\lambda} \varepsilon_+ (\lambda^2 - \lambda^{-2} - |z|^2). \quad (\text{C23})$$

The twisted SU(2) transformation $v^{(\ell)}$ is given by $v^{(\ell)} = (^{(v)}\ell)^{-1} v \ell$ and its infinitesimal version is

$$v^{(\ell)} = \mathbf{1} + i\vec{\varepsilon}' \cdot \vec{\sigma} \quad \text{with } \epsilon'_z = \epsilon_z + \frac{z\varepsilon_- + \bar{z}\varepsilon_+}{2\lambda}, \quad \epsilon'_+ = \lambda^{-2} \varepsilon_+. \quad (\text{C24})$$

That $v^{(\ell)}$ is not simply obtained from v by conjugation by ℓ comes from the fact that triangular matrices are not stable under conjugation by SU(2) matrices and vice-versa. This allows to recover the components of $\delta_L^{(v)} u \sim i(\vec{\varepsilon}' \cdot \vec{\sigma})u$.

$$\begin{aligned} \delta_L^{(v)} \alpha &= i(\varepsilon'_z \alpha + \varepsilon'_- \beta) = i(\varepsilon_z \alpha + \frac{1}{2} \lambda^{-1} (\varepsilon_- z + \varepsilon_+ \bar{z}) \alpha + \lambda^{-2} \varepsilon_- \beta) \\ \delta_L^{(v)} \beta &= i(-\varepsilon'_z \beta + \varepsilon'_+ \alpha) = i(-\varepsilon_z \beta - \frac{1}{2} \lambda^{-1} (\varepsilon_- z + \varepsilon_+ \bar{z}) \beta + \lambda^{-2} \varepsilon_+ \alpha) \end{aligned} \quad (\text{C25})$$

We can recover these transformations thanks to the Poisson brackets given in Proposition II.5 by brute force. We first recall that

$$\kappa^{-1} \text{Tr } V L = \kappa^{-1} (2\epsilon_z \lambda^2 + \epsilon_- \lambda z + \epsilon_+ \lambda \bar{z}) = (2\epsilon_z (T_0 + T_z) + \epsilon_- T_+ + \epsilon_+ T_-). \quad (\text{C26})$$

We have then from Proposition II.5

$$\delta_L^{(v)} \ell = -\lambda^{-2} \{ \kappa^{-1} \text{Tr } V L, \ell \}, \quad \delta_L^{(v)} u = -\lambda^{-2} \{ \kappa^{-1} \text{Tr } V L, u \}, \quad (\text{C27})$$

and by considering each component, we prove they match (C23) and (C25).

$$\delta_L^{(v)} z = -\kappa^{-1} \lambda^{-2} \{ 2\varepsilon_z \lambda^2 + \varepsilon_- \lambda z + \varepsilon_+ \lambda \bar{z}, z \} = -2i \varepsilon_z z + i \varepsilon_+ \lambda^{-1} (\lambda^2 - \lambda^{-2} - \frac{1}{2} |z|^2) - i \varepsilon_- \lambda^{-1} z^2 \quad (\text{C28})$$

$$\delta_L^{(v)} \lambda = -\lambda^{-2} \kappa^{-1} \{ 2\varepsilon_z \lambda^2 + \varepsilon_- \lambda z + \varepsilon_+ \lambda \bar{z}, \lambda \} = \frac{1}{\kappa \lambda^2} (\varepsilon_- \{ \lambda, \lambda z \} + \varepsilon_+ \{ \lambda, \lambda \bar{z} \}) = \frac{i}{2} (\varepsilon_- z - \varepsilon_+ \bar{z}), \quad (\text{C29})$$

$$\delta_L^{(v)} \alpha = -\kappa^{-1} \lambda^{-2} \{ 2\varepsilon_z \lambda^2 + \varepsilon_- \lambda z + \varepsilon_+ \lambda \bar{z}, \alpha \} = i(\varepsilon_z \alpha + \frac{1}{2} \lambda^{-1} (\varepsilon_- z + \varepsilon_+ \bar{z}) \alpha + \lambda^{-2} \varepsilon_- \beta) \quad (\text{C30})$$

$$\delta_L^{(v)} \beta = -\kappa^{-1} \lambda^{-2} \{ 2\varepsilon_z \lambda^2 + \varepsilon_- \lambda z + \varepsilon_+ \lambda \bar{z}, \beta \} = i(-\varepsilon_z \beta - \frac{1}{2} \lambda^{-1} (\varepsilon_- z + \varepsilon_+ \bar{z}) \beta + \lambda^{-2} \varepsilon_+ \alpha) \quad (\text{C31})$$

The right rotations take a simple form as one can see in (C17). Their infinitesimal version is therefore given by

$$\delta_R^{(v)} \ell = 0, \quad \delta_R^{(v)} u = iu(\vec{\varepsilon} \cdot \vec{\sigma}). \quad (\text{C32})$$

We want to show how to recover these expressions from the Poisson brackets in (II.5). The action on the ℓ variable is direct since $\{\tilde{\ell}, \ell\} = 0$.

$$\delta_R^{(v)} \ell = \frac{\tilde{\lambda}^2}{\kappa} \{\text{Tr } V(\tilde{L}^{op})^{-1}, \ell\} = 0. \quad (\text{C33})$$

We calculate the action on u component by component, using (B12).

$$\delta_R^{(v)} u = \frac{\tilde{\lambda}^2}{\kappa} \{\text{Tr } V(\tilde{L}^{op})^{-1}, u\} \rightsquigarrow \begin{cases} \delta_R^{(v)} \alpha = i(\varepsilon_z \alpha - \varepsilon_+ \bar{\beta}) \\ \delta_R^{(v)} \beta = i(\varepsilon_z \beta + \varepsilon_+ \bar{\alpha}) \end{cases} \rightsquigarrow \delta_R^{(v)} u = iu(\vec{\varepsilon} \cdot \vec{\sigma}). \quad (\text{C34})$$

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- [1] S. Carlip, *Quantum gravity in 2+1 dimensions*, Cambridge, UK: Univ. Pr. (1998)
 - [2] L. Freidel and D. Louapre, *Ponzano-Regge model revisited II: Equivalence with Chern-Simons*, gr-qc/0410141.
 - [3] K. Noui, *Three Dimensional Loop Quantum Gravity: Particles and the Quantum Double*, J. Math. Phys. **47** (2006) 102501 [gr-qc/0612144].
 - [4] C. Meusburger and K. Noui, *The Hilbert space of 3d gravity: quantum group symmetries and observables*, Adv. Theor. Math. Phys. **14** (2010) 1651 [arXiv:0809.2875 [gr-qc]].
 - [5] K. Noui and A. Perez, *Three-dimensional loop quantum gravity: Physical scalar product and spinfoam models*, Class. Quant. Grav. **22** (2005) 1739 [gr-qc/0402110].
 - [6] E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, Nucl. Phys. B **311** (1988) 46.
 - [7] V. G. Turaev and O. Y. Viro, *State sum invariants of 3 manifolds and quantum 6j symbols*, Topology **31** (1992) 865.
 - [8] J. Roberts, *Skein theory and Turaev-Viro invariants*, Topology, **34**, 4, (1995) 771-787.
 - [9] K. Noui, A. Perez and D. Pranzetti, *Canonical quantization of non-commutative holonomies in 2+1 loop quantum gravity*, JHEP **1110** (2011) 036. [arXiv:1105.0439 [gr-qc]].
 - [10] M. Dupuis and F. Girelli, *Quantum hyperbolic geometry in loop quantum gravity with cosmological constant*, Phys. Rev. D **87** (2013) 121502 [arXiv:1307.5461 [gr-qc]].
 - [11] M. Dupuis and F. Girelli, *Observables in loop quantum gravity with a cosmological constant*, [arXiv:1311.6841 [gr-qc]].
 - [12] V. Bonzom and L. Freidel, *The Hamiltonian constraint in 3d Riemannian loop quantum gravity*, Class. Quant. Grav. **28** (2011) 195006 [arXiv:1101.3524 [gr-qc]].
 - [13] V. Bonzom and E. R. Livine, *A New Hamiltonian for the Topological BF phase with spinor networks*, arXiv:1110.3272 [gr-qc].
 - [14] S.A. Frolov, *Gauge-invariant Hamiltonian formulation of lattice Yang-Mills theory and the Heisenberg double*, Mod.Phys.Lett. A10 (1995) 2619-2632 [arXiv:hep-th/9501143].
 - [15] S.A. Frolov, *Hamiltonian lattice gauge models and the Heisenberg double*, Mod.Phys.Lett. A10 (1995) 2885-2896 [arXiv:hep-th/9502121].
 - [16] Y. Kosmann-Schwarzbach, *Lie bialgebras, Poisson Lie groups and dressing transformations*. (1997). Lecture Notes in Physics **495**, (1997), 104-170.
 - [17] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge, UK: Univ. Pr. (1994)
 - [18] J. H. Lu, *Multiplicative and Affine Poisson Structures on Lie Groups*, PhD thesis, University of California, Berkeley, 1990.
 - [19] G. Marmo, A. Simoni and A. Stern, "Poisson lie group symmetries for the isotropic rotator," Int. J. Mod. Phys. A **10** (1995) 99 [hep-th/9310145].
 - [20] A. Stern and I. Yakushin, "Deformation quantization of the isotropic rotator," Mod. Phys. Lett. A **10** (1995) 399 [hep-th/9312125].
 - [21] V. V. Fock and A. A. Rosly, *Poisson structure on moduli of flat connections on Riemann surfaces and r matrix*, Am. Math. Soc. Transl. **191** (1999) 67 [math/9802054 [math-qa]].
 - [22] C. Meusburger and B. J. Schroers, *Quaternionic and Poisson-Lie structures in 3d gravity: The Cosmological constant as deformation parameter*, J. Math. Phys. **49** (2008) 083510 [arXiv:0708.1507 [gr-qc]].
 - [23] B. J. Schroers, *Quantum gravity and non-commutative spacetimes in three dimensions: a unified approach*, Acta Phys. Polon. Supp. **4** (2011) 379 [arXiv:1105.3945 [gr-qc]].
 - [24] L. Freidel and S. Speziale, *Twisted geometries: A geometric parametrisation of SU(2) phase space*, Phys. Rev. D **82**, 084040 (2010).
 - [25] F. Girelli and E. R. Livine, *Reconstructing quantum geometry from quantum information: Spin networks as harmonic oscillators*, Class. Quant. Grav. **22** (2005) 3295 [gr-qc/0501075].
 - [26] L. Freidel and E. R. Livine, *The Fine Structure of SU(2) Intertwiners from U(N) representations*, J. Math. Phys. **51** (2010) 082502 [arXiv:0911.3553 [gr-qc]].

- [27] M. Dupuis, S. Speziale and J. Tambornino, *Spinors and Twistors in Loop Gravity and spinfoams*, arXiv:1201.2120 [gr-qc].
- [28] L. Freidel and S. Speziale, *From twistors to twisted geometries*, Phys. Rev. D **82** (2010) 084041.
- [29] M. Bangoura, Y. Kosmann-Schwarzbach, *The Double of a Jacobian Quasi-Bialgebra*, Letters in Mathematical Physics **28**, 13-29, 1993.
- [30] D. Yetter, *Generalized Barrett-Crane Vertices and Invariants of Embedded Graphs*, arXiv:math/9801131 [math.QA]
- [31] K. Noui and P. Roche, *Cosmological deformation of Lorentzian spinfoam models*, Class. Quant. Grav. **20** (2003) 3175 [gr-qc/0211109].
- [32] W. J. Fairbairn and C. Meusburger, *Quantum deformation of two four-dimensional spinfoam models*, J. Math. Phys. **53** (2012) 022501 [arXiv:1012.4784 [gr-qc]].
- [33] M. Han, *4-dimensional Spin-foam Model with Quantum Lorentz Group*, J. Math. Phys. **52** (2011) 072501.